Calculation of Detection and False Alarm Probabilities in Spectrum Pooling Systems

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Abstract— **The innovative new strategy of spectrum pooling enables public access to spectral ranges of already licensed yet rarely used frequency bands by overlaying a secondary mobile radio system (the rental system, RS) to an existing one (the licensed system, LS). Coexistence of both systems is realized by filling the idle time-frequency gaps of the LS. A key issue in spectrum pooling is the reliable and efficient detection of those spectral ranges that are currently accessed by the LS as those ranges have to be spared from the RS's transmission power. In this letter, formulas for the calculation of the detection and false alarm probability are derived for the general case of an arbitrary measurement covariance matrix, allowing for a maximum exploitation of the proposed distributed detection approach.**

Index Terms— **Distributed detection, OFDM overlay, spectrum pooling.**

I. Introduction

PUBLIC mobile radio spectrum is a scarce resource while wide spectral ranges of already licensed frequency bands are only rarely used. The innovative new strategy of spectrum pooling, first mentioned in [1], enables public access to these spectral ranges by overlaying a secondary mobile radio system (the rental system, RS) to an existing one (the licensed system, LS) without requiring any changes to the latter, thus economically enhancing overall spectral efficiency. Coexistence of both systems is realized by filling the time-frequency gaps of the LS during its idle periods while spectral ranges that are currently accessed by licensed users (LUs) are spared from the RS's transmission power. The state of the art in spectrum pooling has recently been presented in [2], showing that an RS based on OFDM modulation is feasible by adaptively leaving a set of subcarriers unmodulated, thus providing a flexible spectral shape that can fill the idle gaps. However, a key issue is the reliable detection of those spectral ranges that are currently accessed by LUs. A high detection probability must be achieved as the amount of interference that the LS encounters from the RS is directly linked to the detection probability. The detection process has to be repeated periodically at time intervals that are short enough to guarantee an upper bound interference duration at the beginning of an LU's access. At the same time, the detection duration and the false alarm probability should remain as low as possible for the sake of the RS's efficiency. The detection process results in a binary allocation vector, indicating those OFDM subcarriers that have to be spared from the RS's transmission power.

In [3], a distributed detection approach is proposed where any associated rental user (RU) conducts its own detection. The individually decided allocation vectors are then transmitted to a central access point (AP) using a highly efficient signaling scheme called boosting protocol [4]. At the AP, the final decision is made and the resulting allocation vector is broadcasted back to the RUs. There, the PHY layers are then configured with a common subset of OFDM subcarriers that are to be used until the next detection cycle. The resulting diversity of this distributed detection approach yields dramatic improvements in terms of false alarm and detection probability compared to the case of only one station conducting spectral measurements. However, the full advantage of this approach can only be achieved by adapting the individual detection probability to the number of currently involved RUs. This is necessary in order to maintain a specified overall detection probability P_D^{RS} at the AP after the final decision is made. As the number of currently associated RUs is known to the AP, the required detection probability P_D for an individual detection process of one RU can be calculated and broadcasted. Hence, at every RU the detection threshold that realizes the required detection probability has to be calculated. This threshold value has yet only been calculated for the special cases of uncorrelated measurements and fully correlated measurements [3]. In the following, a calculation formula for the general case of an arbitrary measurement covariance matrix is derived, based on the optimal detection rule.

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II. The Detection Model

In this section the detection model is illustrated in brief. Under the worst case assumption of a non line of sight (NLOS) connection between an LU and an RU, the receive signal at the detecting RU can be considered a zero-mean rotationally symmetrical complex Gaussian process according to the central limit theorem. At the receiver, the signal from the LU is disturbed by thermal and background noise. These noise sources are assumed to be white, zero-mean and Gaussian. The resulting process is blockwise transformed into the frequency domain by the immanently available FFT of the OFDM receiver. The consecutively arriving frequency samples corresponding to the useful signal of one LU's subband can be combined in a vector $\mathbf{z}$, containing the real and imaginary parts $\mathbf{x}$, $\mathbf{y}$ of the respective FFT bins. As an FFT is a linear operation, it can be shown that $\mathbf{z}$ still has a normal distribution. Let n denote the number of FFT operations performed during a detection process and m the width of an LU's subband in OFDM subcarriers. Then the probability density function (pdf) of $\mathbf{z}$ can be written as

$$f_S(\mathbf{z}) = \frac{1}{\sqrt{(2\pi)^{2nm}\det \mathbf{C_{SS}}}} \exp\left(-\frac{1}{2}\mathbf{z}^T {\mathbf{C_{SS}}}^{-1}\mathbf{z}\right) \quad (1)$$

with

$$\mathbf{z} = (\mathbf{x}, \mathbf{y})^T = \big(\underbrace{x_{1,1}, \ldots, x_{n,m}}_{nm \text{ real parts}}, \underbrace{y_{1,1}, \ldots, y_{n,m}}_{nm \text{ imag. parts}}\big)^T, \quad (2)$$

where $\mathbf{C_{SS}}$ represents the nonsingular covariance matrix of the time-frequency samples of the considered LU. Due to the lack of synchronization between the two systems, the stochastic properties of real and imaginary parts are identical. Hence, $\mathbf{C_{SS}}$ can be expressed by a symmetrical block matrix

$$\mathbf{C_{SS}} = \begin{pmatrix} \mathbf{C_{XX}} & \mathbf{C_{XY}} \\ \mathbf{C_{XY}} & \mathbf{C_{XX}} \end{pmatrix} \in \mathbf{R}^{(2nm)\times(2nm)}, \quad (3)$$

where the diagonal elements, namely σ_S^2, are the mean receive power of the real and imaginary parts. Because of the whiteness assumption, the corresponding time-frequency samples of the noise process are distributed according to

$$f_N(\mathbf{z}) = \frac{1}{(2\pi\sigma_N^2)^{nm}} \exp\left(-\frac{\mathbf{z}^T\mathbf{z}}{2\sigma_N^2}\right), \quad (4)$$

where σ_N^2 is just the mean noise power of the real and imaginary parts. As the LU signal is additively disturbed by the noise process, the pdf of the resulting samples can be calculated by convolution of $f_S(\mathbf{z})$ and $f_N(\mathbf{z})$, yielding the conditional pdfs

$$f_{R|\text{no LU}}(\mathbf{z}|\text{no LU}) = f_N(\mathbf{z}) \quad (5)$$

$$f_{R|\text{LU}}(\mathbf{z}|\text{LU}) = \frac{\exp\left(-\frac{1}{2}\mathbf{z}^T(\mathbf{C_{SS}} + \sigma_N^2\mathbf{I})^{-1}\mathbf{z}\right)}{\sqrt{(2\pi)^{2nm}\det(\mathbf{C_{SS}} + \sigma_N^2\mathbf{I})}}. \quad (6)$$

The optimal detection rule that classifies whether or not an LU access has occurred in the considered subband is based on the well known Neyman-Pearson criterion [5] that maximizes the detection probability P_D at a given false alarm probability P_F or minimizes P_F at a given P_D, respectively. Applying this criterion yields

$$P_F = \int_{\mathbf{G}} f_{R|\text{no LU}}(\mathbf{z}|\text{no LU})\,\mathrm{d}\mathbf{z} \quad (7)$$

$$P_D = \int_{\mathbf{G}} f_{R|\text{LU}}(\mathbf{z}|\text{LU})\,\mathrm{d}\mathbf{z}, \quad (8)$$

where $\mathbf{G} \subset \mathbf{R}^{(2nm)\times(2nm)}$ is the area that contains all vectors $\mathbf{z}$ leading to the decision that an LU access has occurred. The optimal decision space $\mathbf{G}$ is obtained from the likelihood ratio

$$\frac{f_{R|\text{LU}}(\mathbf{z}|\text{LU})}{f_{R|\text{no LU}}(\mathbf{z}|\text{no LU})} \overset{\text{LU}}{>} \gamma \quad (9)$$

where the choice of γ determines the detection probability P_D. Applying (5) and (6) to (9) yields the optimal decision rule

$$\mathbf{z}^T\left((\sigma_N^2\mathbf{I})^{-1} - (\mathbf{C_{SS}} + \sigma_N^2\mathbf{I})^{-1}\right)\mathbf{z} \overset{\text{LU}}{>} 2\ln\left(\gamma\sqrt{\frac{\det(\mathbf{C_{SS}} + \sigma_N^2\mathbf{I})}{(\sigma_N^2)^{2nm}}}\right), \quad (10)$$

where the right hand side of (10) can be combined to a new threshold value u. All vectors $\mathbf{z}$ fulfilling inequality (10) are assigned to $\mathbf{G}$. The calculation of P_D, P_F against the threshold u requires solving the multidimensional integrals in (8), (7) with respect to the integration region $\mathbf{G}$ given by (10). Although this task seems quite demanding for general $\mathbf{C_{SS}}$, the problem can be analytically reduced into a convenient format by successively applying linear transformations [6].

III. Calculation of the Detection Probability

For the sake of clarity, the following abbreviations are introduced

$$\mathbf{V} = \mathbf{C_{SS}} + \sigma_N^2\mathbf{I} \quad (11)$$

$$\mathbf{A} = (\sigma_N^2\mathbf{I})^{-1} - (\mathbf{C_{SS}} + \sigma_N^2\mathbf{I})^{-1} = (\sigma_N^2\mathbf{I})^{-1} - (\mathbf{V})^{-1} \quad (12)$$

$$u = 2\ln\left(\gamma\sqrt{\frac{\det(\mathbf{C_{SS}} + \sigma_N^2\mathbf{I})}{(\sigma_N^2)^{2nm}}}\right) \quad (13)$$

with $\mathbf{V}$, $\mathbf{A}$ being symmetrical and positive definite matrices. Rewriting (8) yields

$$P_D = \int\limits_{\mathbf{z}^T\mathbf{A}\mathbf{z}>u} \frac{\exp\left(-\frac{1}{2}\mathbf{z}^T\mathbf{V}^{-1}\mathbf{z}\right)}{(2\pi)^{\frac{1}{2}2nm}\det(\mathbf{V})^{\frac{1}{2}}}\,\mathrm{d}\mathbf{z}. \quad (14)$$

$\mathbf{V}$ can be factorized into $\mathbf{V} = \mathbf{L}\mathbf{L}^T$ with a nonsingular lower triangular matrix $\mathbf{L}$ using Cholesky decomposition. Substituting $\mathbf{a} = \mathbf{L}^{-1}\mathbf{z}$ simplifies (14) into

$$P_D = \int\limits_{\mathbf{a}^T(\mathbf{L}^T\mathbf{A}\mathbf{L})\mathbf{a}>u} \frac{\exp\left(-\frac{1}{2}\mathbf{a}^T\mathbf{a}\right)}{(2\pi)^{\frac{1}{2}2nm}}\,\mathrm{d}\mathbf{a}, \quad (15)$$

as $\det(\mathbf{V}) = \det(\mathbf{L})^2$ and $\mathrm{d}\mathbf{z} = \det(\mathbf{L})\,\mathrm{d}\mathbf{a}$. The regular matrix $\mathbf{L}^T\mathbf{A}\mathbf{L}$ can be diagonalized with an eigen decomposition into $\mathbf{L}^T\mathbf{A}\mathbf{L} = \mathbf{T}\mathbf{D}\mathbf{T}^{-1}$, where $\mathbf{D}$ is the diagonal matrix of eigenvalues of $\mathbf{L}^T\mathbf{A}\mathbf{L}$ and $\mathbf{T}$ the associated matrix of eigenvectors. Since $\mathbf{L}^T\mathbf{A}\mathbf{L}$ is also symmetrical, $\mathbf{T}$ is an orthonormal matrix, thus $\mathbf{T}^{-1} = \mathbf{T}^T$ and $\det(\mathbf{T}) = 1$. This yields $\mathbf{T}^T(\mathbf{L}^T\mathbf{A}\mathbf{L})\mathbf{T} = \mathbf{D}$, which gives rise to another substitution $\mathbf{b} = \mathbf{T}^T\mathbf{a}$. Since $\mathbf{b}^T\mathbf{D}\mathbf{b} = \mathbf{a}^T\mathbf{T}\mathbf{T}^T(\mathbf{L}^T\mathbf{A}\mathbf{L})\mathbf{T}\mathbf{T}^T\mathbf{a} = \mathbf{a}^T(\mathbf{L}^T\mathbf{A}\mathbf{L})\mathbf{a}$, applying this second substitution to (15) results in

$$P_D = \int\limits_{\mathbf{b}^T\mathbf{D}\mathbf{b}>u} \frac{\exp\left(-\frac{1}{2}\mathbf{b}^T\mathbf{b}\right)}{(2\pi)^{\frac{1}{2}2nm}}\,\mathrm{d}\mathbf{b} = 1 - \int\limits_{\mathbf{b}^T\mathbf{D}\mathbf{b}\leq u} \frac{\exp\left(-\frac{1}{2}\mathbf{b}^T\mathbf{b}\right)}{(2\pi)^{\frac{1}{2}2nm}}\,\mathrm{d}\mathbf{b}. \quad (16)$$

The matrices $\mathbf{L}^T\mathbf{A}\mathbf{L}$ and $\mathbf{V}\mathbf{A}$ are similar, which can be easily seen regarding the similarity transformation $\mathbf{L}(\mathbf{L}^T\mathbf{A}\mathbf{L})\mathbf{L}^{-1} = \mathbf{L}\mathbf{L}^T\mathbf{A} = \mathbf{V}\mathbf{A}$ with the invertible transformation matrix $\mathbf{L}$. Therefore, they have the same eigenvalues λ_i. Inserting (11), (12) back into $\mathbf{V}\mathbf{A}$ and simplifying the expression ends up with the very simple result

$$\mathbf{V}\mathbf{A} = \frac{\mathbf{C_{SS}}}{\sigma_N^2}. \quad (17)$$

Thus, the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2nm} > 0$ building up the diagonal matrix $\mathbf{D}$ in (16) are just the eigenvalues of the normalized (with respect to σ_N^2) covariance matrix of the time-frequency samples of the LU.

Noting that $\mathbf{b}^T\mathbf{D}\mathbf{b} = \sum_{i=1}^{2nm}\lambda_i b_i^2$, the integral in (16) can be interpreted as the cumulative distribution function (cdf) of a sum of weighted χ^2 variables

$$P_D = 1 - \Pr\left\{\sum_{i=1}^{2nm}\lambda_i B_i^2 \leq u\right\}, \quad (18)$$

where the $B_i \sim \mathcal{N}(0,1)$ are i.i.d unit normal variables. Quite a lot of attention has been paid to the problem of numerically evaluating the probability of a weighted sum of χ^2 variables since the computation of the exact probability can be regarded

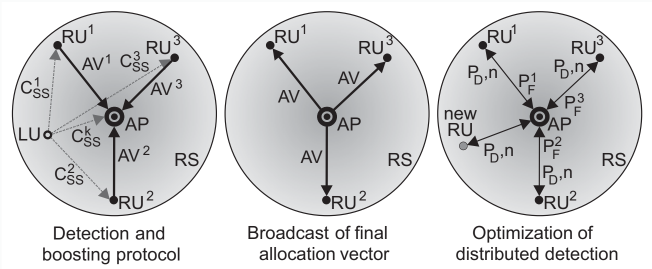

Fig. 1. Information flow within the distributed detection scheme.

as very complicated. In [7], an efficient yet extremely accurate approximation is presented. The algorithm assumes $\lambda_i > 0$ and $\sum \lambda_i = 1$. To comply with these requirements, the weights in (18) need to be scaled. Let $d_i = \lambda_i / \sum \lambda_i$ and $w = u / \sum \lambda_i$, then

$$P_D = 1 - \Pr\left\{\sum_{i=1}^{2nm} d_i B_i^2 \leq w\right\} \text{ with } \sum_{i=1}^{2nm} d_i = 1. \quad (19)$$

The desired probability is then approximated by

$$P_D = 1 - \min\{G(w), H(w)\} \text{ with} \quad (20)$$

$$G(w) = \sum_{i=1}^{2nm} d_i \frac{\int_0^{\frac{w}{2d_i}} t^{\frac{1}{2d_i}-1} e^{-t}\, \mathrm{d}t}{\Gamma(\frac{1}{2d_i})} \quad (21)$$

$$H(w) = \Pr\left\{\sum_{i=1}^{2nm} B_i^2 \leq \frac{w}{\delta}\right\} = \frac{\int_0^{\frac{w}{\delta}} t^{nm-1} e^{-\frac{t}{2}} \mathrm{d}t}{\Gamma(nm) 2^{nm}} \quad (22)$$

$$\delta = \sqrt[2nm]{\prod_{i=1}^{2nm} d_i}. \quad (23)$$

Since P_D is the given value in the context of the distributed detection of LUs, (20) has to be solved for w, which can be realized using some gradient descent method. For an acceptable efficiency however, the integrals used in (21) and (22) may need to be tabulated.

IV. Calculation of the False Alarm Probability

The evaluation of false alarm probabilities does not necessarily need to be conducted since the main concern is to achieve a specified detection probability. As the width m of an LU's subband is assumed constant for a given LS, the number n of FFT operations could be some preassigned fixed value, large enough to attain an adequate efficiency at some assumed average conditions. However, as there exists an optimal n that maximizes the RS's efficiency η, it is straightforward to consider n as a variable rather than a constant. η is given by the proportion of the remaining idle time-frequency gaps after the detection process:

$$\eta = n \frac{T_{\text{FFT}}}{T_{\text{cycle}}} (1 - P_F^{RS}), \quad (24)$$

where T_{FFT} is the measurement period previous to an FFT (usually the duration of one OFDM symbol), T_{cycle} the time between consecutive detection events and P_F^{RS} the resulting overall false alarm probability at the AP according to the distributed detection scheme. Let k be the number of currently associated RUs and let $j = 1, \ldots, k$ be the index denoting the jth RU. P_F^{RS} implicitly depends on n over the functional dependency

$$P_F^{RS} = 1 - \prod_{j=1}^{k} \left[1 - P_F^j\right] = 1 - \prod_{j=1}^{k} \left[1 - f(P_D, \mathbf{C}_{\mathbf{SS}}^j, n)\right]$$
$$= 1 - \prod_{j=1}^{k} \left[1 - f(1 - \sqrt[k]{1 - P_D^{RS}}, \mathbf{C}_{\mathbf{SS}}^j, n)\right], \quad (25)$$

where f denotes the receiver operating characteristic. Since P_D^{RS}, k and $\mathbf{C}_{\mathbf{SS}}^j$ are externally determined, n is the only parameter that can be changed in order to maximize η using some optimization strategy. For this purpose, P_F^{RS} must be known at the AP. Thus, every RU must calculate and transmit its respective P_F^j as depicted in Fig. 1.

The derivation of P_F can be carried out according to the derivation of P_D in the last section and is omitted here for convenience. The result is identical to (18) with the weights being the eigenvalues of (12) instead of (17). Applying the following rules

$$\lambda \in S\{\mathbf{A}\} \Rightarrow \lambda^{-1} \in S\{\mathbf{A}^{-1}\}, (\alpha\lambda + c) \in S\{\alpha\mathbf{A} + c\mathbf{I}\}, \quad (26)$$

where $S\{\mathbf{A}\}$ denotes the set of eigenvalues of matrix $\mathbf{A}$, P_F can be expressed in terms of eigenvalues of $\frac{\mathbf{C}_{\mathbf{SS}}}{\sigma_N^2}$, yielding

$$P_F = 1 - \Pr\left\{\sum_{i=1}^{2nm} \frac{\lambda_i}{\sigma_N^2(\lambda_i + 1)} B_i^2 \leq u\right\}. \quad (27)$$

Thus, eigenvalues must be calculated only once at every RU.

V. Conclusion

In this letter, formulas for the calculation of the detection and false alarm probability in a spectrum pooling system have been derived for the general case of an arbitrary LU's covariance matrix. Within the distributed detection approach proposed in [3], these formulas allow for an optimized detection algorithm that adaptively maximizes the RS's efficiency under changing conditions of k and $\mathbf{C}_{\mathbf{SS}}^j$ while still meeting the constraint of a mandatory overall detection probability. Every associated RU has to estimate the covariance matrices of the currently accessing LUs (with respect to the broadcasted allocation vector), calculate the corresponding eigenvalues and set the detection thresholds u according to the detection probability required by the AP. After the RUs have transmitted their resulting false alarm probabilities to the AP, n can be adapted in order to maximize the RS's efficiency. The interesting problem of how to estimate the covariance matrices and how to adapt n within the resulting feedback loop in an optimal fashion is beyond the scope of this letter and needs to be further investigated.

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