Spectrum Sharing for Unlicensed Bands

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***Abstract*— We study a spectrum sharing problem in an unlicensed band where multiple systems coexist and interfere with each other. Due to asymmetries and selfish system behavior, unfair and inefficient situations may arise. We investigate whether efficiency and fairness can be obtained with self-enforcing spectrum sharing rules. These rules have the advantage of not requiring a central authority that verifies compliance to the protocol.**

Any self-enforcing protocol must correspond to an equilibrium of a game. We first analyze the possible outcomes of a one shot game, and observe that in many cases an inefficient solution results. However, systems often coexist for long periods and a repeated game is more appropriate to model their interaction. In this repeated game the possibility of building reputations and applying punishments allows for a larger set of self-enforcing outcomes. When this set includes the optimal operating point, efficient, fair, and incentive compatible spectrum sharing becomes possible. We present examples that illustrate that in many cases the performance loss due to selfish behavior is small. We also prove that our results are tight and quantify the best achievable performance in a non-cooperative scenario.

***Index Terms*— Game theory, Gaussian interference game, interference channel, Nash equilibrium, resource allocation, self-enforcing protocols, spectrum sharing.**

I. Introduction

We STUDY a scenario where multiple wireless systems share the same spectrum. For concreteness consider a typical urban area with 802.11 networks, bluetooth systems, walkie-talkies, etc. co-existing and operating in the same unlicensed band, e.g. ISM, UNII, etc. The systems do not have a common goal and do not cooperate with each other. We assume that spectrum is a scarce resource, so that efficiency is a concern. We are interested in designing *spectrum sharing rules* and protocols which allow the systems to share the bandwidth in a way that is *fair*, *efficient* and *compatible* with the *incentives* of the individual systems.

A resource allocation is *efficient* if it is not possible to improve the performance of a given system without degrading the performance of some other system. Usually, there are many efficient operating points, each representing a different performance trade-off among the systems. *Fairness* is related to the relative performance among the systems. It can be achieved by optimizing a global utility function over the possible resource allocations. Different utilities represent different fairness goals. Finally, an allocation is *incentive compatible* or *self-enforcing* if there is no incentive for an individual system to deviate from it.

In order to attain efficient resource allocations in diverse situations the spectrum sharing rules must be flexible. The rules should adapt to each specific scenario, taking into account the number of systems sharing the spectrum in a given time and location, and considering the particular interaction between the systems. However, adding flexibility to selfish systems may lead to inefficient and unfair situations. For example, a greedy system may not have an incentive to follow the spectrum sharing rules, and as a result may use more resources (e.g. power, bandwidth) than allowed.

The issue of incentives will play a central role in our analysis. We assume that the resource allocation is determined as the outcome of a game. Each system calculates a set of operating points which are Nash equilibria of the game. When there is more than one Nash Equilibrium point, the systems follow a convention (this could be widely known standard or an FCC mandate) to pick one of these points. Since the systems operate at a Nash equilibrium there is no incentive for any individual system to deviate from this point.

The key problem is therefore to design spectrum sharing rules which lead to a Nash equilibrium that is fair and efficient. These rules are *self enforcing*, and as a result do not require the intervention of an external authority to verify compliance. This is of particular importance with the advent of new technologies like software-defined radios, which are inherently hard to certify and easy to alter.

We start by formulating a one shot game in which each system chooses its power allocation once and for all, and this yields the data rates at the operating point. We find, by extending a result of [5] that in low interference situations, the full-spread equilibrium is the only possible outcome of the game. And in many cases, the rates that result from the full-spread equilibrium are suboptimal (inefficient, unfair, or both). This is a negative result from the point of view of designing a standard, as it would be desirable to have multiple equilibrium points to choose from.

However, systems operate and have to co-exist over a long period of time, and in this context, it may be more reasonable to model the scenario as a repeated game where systems play multiple rounds, remembering past experiences. The situation brightens considerably once one considers repeated games. We show that in a repeated game, any vector of rates in the achievable region that is component-wise larger than the full-spread rates (obtained when all systems spread their power uniformly over frequency) can be supported. Therefore, if the optimal rate vector (efficient and fair according to some global objective) is component-wise greater than the full-spread rate vector, there is no performance loss due to lack of cooperation.

In essence, the systems can enforce any such operating point by threatening to apply a punishment if any individual system deviates from the power allocation that achieves the desired operating point. The punishment is simply to apply the power allocation at the Nash equilibrium of the static game whose rates are component-wise smaller.

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The computation of the fair and efficient operating point and the choice of strategies that achieve it require common knowledge of all the parameters (channel gains, power constraints, etc.). These parameters have to be measured and communicated among the systems. A selfish system may have an incentive to falsify some of its parameters in order to obtain an advantage over the other systems. We show that in most cases, dishonest behaviors can be detected and punished. Therefore, by adding punishment to the parameter measurement process we can incentivize the systems to behave truthfully.

In our work we provide a unified framework to study the issues of efficiency, fairness and incentive compatibility in a non-cooperative spectrum sharing situation. Individual aspects of the problem have been considered separately in many related papers. For example, [5] proposes the iterative waterfilling algorithm (IWA) to obtain good power spectral allocations for spectrum sharing in a cooperative setting. Sufficient conditions for the uniqueness of the equilibrium of the IWA (which coincide with the Nash equilibria of the Gaussian Interference Game to be discussed in Section IV) have been presented in [19]. [12] observes that iterative waterfilling may lead to inefficient solutions. [18] overcomes some of the difficulties of IWA by exchanging "interference prices" to take into account the interference created onto other systems. Obtaining efficient and fair allocations requires solving optimization problems. The problem of optimizing resource allocations has been studied in [14], [15], [16]. Many of these works rely on relaxations to solve the optimization with tractable complexity. [13] studies a repeated game between selfish players in a wireless model. It uses a genetic algorithm to find *good* strategies in a limited strategy space. This analysis, however, does not consider efficiency and fairness issues, and the resulting strategies may not be incentive compatible. [17] considers the issues of fairness and efficiency in non-cooperative wireless applications, and proposes the use of punishment to achieve a desired operating point. The games considered in [17], however, are games of complete information with small strategy spaces, and are very different from the games considered here.

This rest of this work is organized as follows. In Section II we present the model to be used in the following sections. Section III focuses on the issues of fairness and efficiency in an ideal situation where systems cooperate with each other. Section IV analyzes non-cooperative situations. Section V considers the problem of channel measurement and exchange between possibly dishonest systems. Finally Section VI presents some conclusions and open problems. Appendix A introduces a more general model than that of Section II and shows that our main results hold in a broader setting where we allow the systems to use arbitrary codes (i.e. non-Gaussian). In addition, for clarity of presentation some results are proved in the Appendices.

Preliminary versions of this article have been presented in [1] and [2].

II. Channel Model

We model a situation in which M systems, each formed by a single transmitter-receiver pair, coexist in the same area. Consider an M user Gaussian interference channel in discrete time defined by:

$$y_i[n] = \sum_{j=1}^{M} h_{j,i} x_j[n] + z_i[n];\ \text{i=1,\dots,M} \tag{1}$$

where $x_i, y_i, z_i \in \mathbb{C}$ and the noise processes are i.i.d. over time with $z_i \sim \mathcal{CN}(0, N_0)$. By assuming that the channel from each transmitter to each receiver has a single tap we are restricting attention to the case of flat fading. The input of user i has an average power constraint P_i.

We will assume that each system treats the received interference as noise. This leads to a tractable inner bound to the capacity region of the interference channel. In addition, practical limitations such as decoder complexity, uncertainty in the estimation of $\{h_{j,i}\}$, delay constraints, etc., may preclude the use of interference cancellation techniques. Therefore the assumption of treating interference as noise may be realistic in many cases.

Finally, we will assume that the systems use random Gaussian codebooks, which means that the transmitted signals look like white Gaussian processes. In Appendix A we consider a model where the codebooks of the systems are allowed to be non-Gaussian, and show that most of our results extend to this scenario.

Under these assumptions, using the capacity expression for the single user Gaussian channel, we can determine the maximum rate that system i can achieve for specific power allocations[1]:

$$R_i = \int_0^W \log\left(1 + \frac{c_{i,i} p_i(f)}{N_0 + \sum_{j \neq i} c_{j,i} p_j(f)}\right) df \tag{2}$$

where $p_i(f)$ is the power spectral density of the input signal of system i, and where for convenience we defined $c_{i,j} = |h_{i,j}|^2$. Note that due to the power constraints, $p_i(f)$ must satisfy:

$$\int_0^W p_i(f) df \leq P_i. \tag{3}$$

The spectrum sharing problem that we consider is to determine a set of power allocations $\{p_i(f)\}$ for the M systems, that maximizes a given global utility function while satisfying the power constraints. This maximization results in allocations that are fair and efficient in a cooperative scenario, i.e. free from the problem of incentives. In the next section we study the structure of the optimal power allocations for any reasonable choice of global utility.

III. Optimal Spectrum Allocations

Many of the fairness issues in spectrum sharing arise due to asymmetries between the systems. Figure 1 shows three different examples where two systems operate in asymmetric situations. In scenario (a) both systems have similar power capabilities (e.g. two 802.11 systems) but due to the locations of the transmitters and receivers, one system receives large interference while the other does not. Scenarios (b) and (c) describe situations where a high power system (e.g. 802.11

[1]In all cases we use $\log(\cdot)$ for a base 2 logarithm.

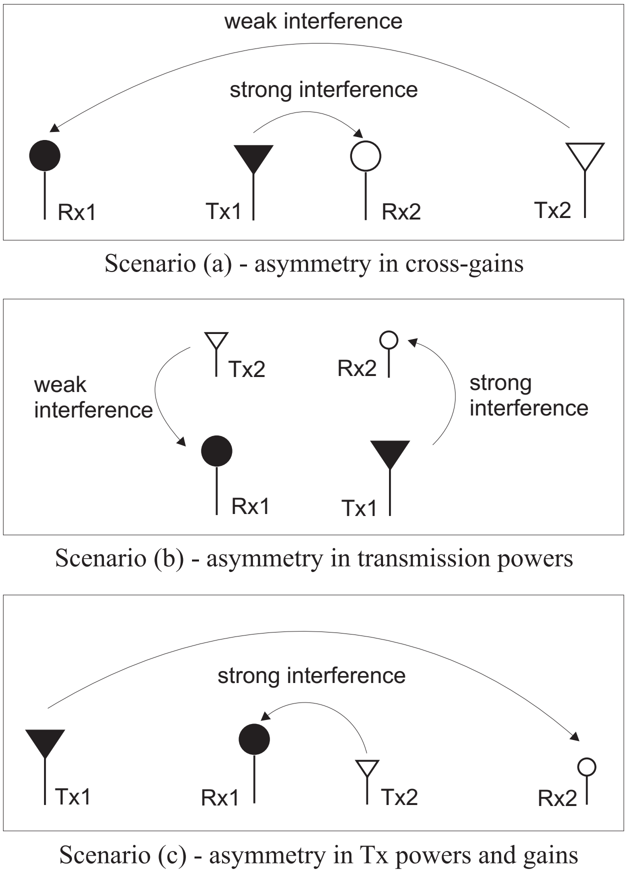

Fig. 1. Three examples of asymmetric situations between two systems sharing the same band. The sizes of the antennas represent power capabilities, and smaller distances indicate higher gains.

system) shares spectrum with a low power system (e.g. bluetooth system). In (b) all the gains are comparable, so intuitively the weak system is in disadvantage. In (c) due to asymmetry in the gains both systems can interfere with each other and one can imagine that a more fair situation may result.

For concreteness we assign specific parameter values to each scenario. Without loss of generality we can assume in all cases that $c_{1,1} = c_{2,2} = 1$, $N_0 = 1$ and $W = 1$. In scenario (a) we choose $P_1 = P_2 = 10$, $c_{1,2} = 10$ and $c_{2,1} = 0.5$. For scenario (b) we set $P_1 = 10$, $P_2 = 1$, and $c_{1,2} = c_{2,1} = 1.1$. Finally in (c) we set $P_1 = 10$, $P_2 = 1$, $c_{1,2} = 0.5$ and $c_{2,1} = 10$.

Imagine that in these three scenarios we want to maximize some global utility function $U(R_1, R_2)$, that represents some fairness objective. We are interested in determining the maximum value of U and the corresponding power spectral allocations that achieve it. In this section we will show how to solve this problem efficiently.

We assume that all the parameters are know to all the systems performing the optimization. In particular, we assume that the number of systems sharing the spectrum is common knowledge. Practical algorithms for the estimation and exchange of parameters are presented in Section V.

Let $\mathcal{R}$ be the achievable rate region:

$$\mathcal{R} = \left\{ \mathbf{R} : R_i = \int_0^W \log\left(1 + \frac{c_{i,i}p_i(f)}{N_0 + \sum_{j\neq i} c_{j,i}p_j(f)}\right) df \right.$$

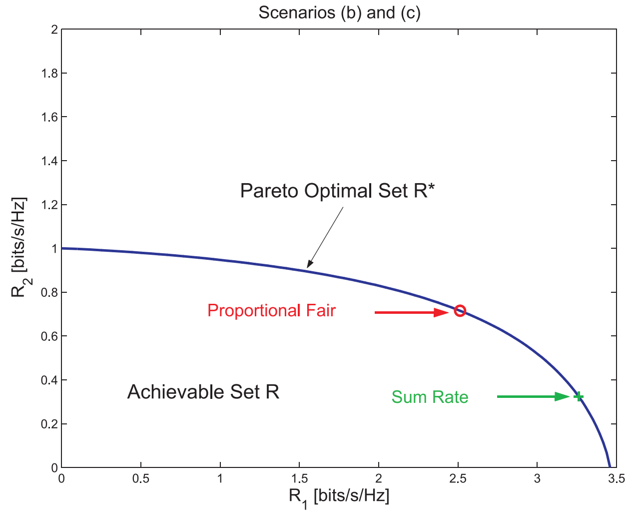

Fig. 2. Achievable set $\mathcal{R}$ and Pareto efficient set $\mathcal{R}^*$ for scenarios (b) and (c).

$$\left. \text{and} \int_0^W p_i(f)df \leq P_i \text{ with } p_i(f) \geq 0 \text{ for } i = 1, \ldots, M \right\} \quad (4)$$

where $\mathbf{R} = (R_1, R_2, \ldots, R_M)$ and let $\mathcal{R}^*$ be the set of Pareto optimal points of $\mathcal{R}$:

$$\mathcal{R}^* = \Big\{ (R_1, \ldots, R_M) \in \mathcal{R} : R_i \geq \tilde{R}_i \ \forall (R_1, \ldots, R_{i-1}, \tilde{R}_i, R_{i+1}, \ldots, R_M) \in \mathcal{R}, \text{ for } i = 1, \ldots, M \Big\}. \quad (5)$$

In words, a rate allocation is Pareto optimal (or efficient) if it is not possible to increase the rate of any system without decreasing the rate of some other system.

Figure 2 shows the achievable set and Pareto optimal set for scenarios (b) and (c). The reason why for this specific choice of parameters in both scenarios we obtain the same sets will be explained later in this section.

The choice of the utility function will strongly influence the fairness in the resulting allocations. For example we may consider $U_{sum}(R_1, R_2) = R_1 + R_2$ if we are interested in maximizing the total sum rate. While in scenario (a) this choice of utility results in an optimal operating point where $R_1 = R_2$, in scenarios (b) and (c) the resulting optimal allocations are very unfair for system 2 ($R_2 \ll R_1$) (see Figure 2). A more fair allocation results from choosing the proportional fair metric $U_{PF}(R_1, R_2) = \log(R_1) + \log(R_2)$ proposed in [10]. By applying the $\log(\cdot)$ function to each rate, we give higher priority to the system in disadvantage. We can see in Figure 2 how in scenarios (b) and (c) the use of the proportional fair metric results in a more fair allocation. Note that in scenario (a) the use of U_{PF} results in the same rates as when U_{sum} is used.

For any utility function that is component-wise monotonically increasing in $(R_1, \ldots, R_M)$, the optimal rate allocation must occur in a point of the boundary $\mathcal{R}^*$. So it is of interest to obtain a simple characterization for $\mathcal{R}$ and $\mathcal{R}^*$.

At first glance, computing $\mathcal{R}$ requires to search over all possible power allocations $p_i(f)$ that satisfy the power constraint. Since $p_i(f)$ are functions with arbitrarily many degrees of freedom, the computation of $\mathcal{R}$ seems to be an infinite dimensional problem. However the following theorem shows that we can restrict attention to piecewise constant power allocations, and as a result, the problem of computing $\mathcal{R}$ has finite dimension.

Theorem 1: Any point in the achievable rate region $\mathcal{R}$ defined in (4) can be obtained with M power allocations that are piecewise constant in the intervals $[0, w_1), [w_1, w_2), \ldots, [w_{2M-1}, W]$, where $w_i \leq w_{i+1}, i = 1, \ldots, 2M-2$, for some choice of $\{w_i\}_{i=1}^{2M-1}$.

Proof: See Appendix B. ∎

Note that once we fix the choice of intervals to obtain a point in $\mathcal{R}$, the M power allocations are constant in the same intervals.

This result arises naturally from geometric considerations in our model where frequency is a continuous variable. In the digital subscriber line (DSL) and wireless communications literature the available bandwidth is often divided into N discrete channels, and frequency is treated as a discrete parameter [5], [16], [18]. This approximation of continuous frequency into discrete channels makes the complexity of the resulting optimization scale with N. In the flat fading case this is an artifact of the discretization of frequency, since the true complexity of the problem is a function of the number of systems M and not the number of channels N.

For the special case of channels satisfying a pairwise high interference condition (which is satisfied with the choice of parameters in scenarios (a), (b) and (c)), it turns out that the optimal power allocations are orthogonal, and hence the characterization of $\mathcal{R}^*$ is further simplified.

Theorem 2: Let $(R_1, \ldots, R_M)$ be a Pareto efficient rate vector achieved with power allocations $\{p_i(f)\}_{i=1,\ldots,M}$. If $c_{i,j}c_{j,i} > c_{i,i}c_{j,j}$ then the power allocations $p_i(f)$ and $p_j(f)$ are orthogonal, i.e. $p_i(f)p_j(f) = 0$ for $f \in [0, W]$.

Proof: See Appendix C. ∎

The condition $c_{i,j}c_{j,i} > c_{i,i}c_{j,j}$ means that for systems i and j, the product of the channel cross gains $c_{i,j}c_{j,i}$ is greater than the product of the channel direct gains $c_{i,i}c_{j,j}$. Note that the condition can be satisfied even if one of the cross gains is small, by having the other cross gain large enough. Also, note that the condition is independent of the power constraints $\{P_i, P_j\}$ and noise variance N_0. It is easy to check that this high interference condition is satisfied in our three examples due to our choice of parameters.

In particular, if $c_{i,j}c_{j,i} > c_{i,i}c_{j,j}$ for any $i \neq j$, $j = 1, \ldots, M$, we can achieve any Pareto efficient rate vector with frequency division multiplexing (FDM). In this case, the maximization of any concave, non-decreasing function of $(R_1, \ldots, R_M)$ becomes a concave optimization problem and can be efficiently solved. In particular, the weighted sum rate utility $U_{WS}(R_1, \ldots, R_M) = \sum_{i=1}^{M} u_i R_i$ for non-negative weights $\{u_i\}_i$, and the proportional fair utility $U_{PF} = \sum_{i=1}^{M} \log(R_i)$ result in concave problems. These results allow to easily compute the rates in Figure 2.

Note that since the Pareto efficient rates are obtained with orthogonal allocations when $c_{1,2}c_{2,1} > 1$ (for direct gains equal to 1), the actual values of the cross gains $c_{1,2}$ and $c_{2,1}$ have no influence on the achievable region. This explains why scenarios (b) and (c) result in the same achievable region and optimal rates.

When the conditions of Theorem 2 are not satisfied, we can use techniques such as Lagrangian methods to solve the problem of maximizing $U(R_1, \ldots, R_M)$ with tractable complexity.

IV. Non-cooperative Scenarios

Throughout the previous section we have implicitly assumed that the M systems cooperate to maximize a global utility function by choosing appropriate power allocations. This assumption may be realistic when the different systems are jointly designed with a common goal, are complying with some standard or regulation, or are in fact transmitter-receiver pairs of a single global system.

However, in a spectrum sharing scenario where regulations may be lax and systems may be competing with one another to gain access to the shared medium, assuming selfish behavior may be more realistic. In this section we analyze how the lack of cooperation among systems may affect the set of achievable rates.

We will consider the same model introduced in Section II under the assumption that the different systems behave selfishly and rationally. We associate to each system i a utility function $U_i(R_i)$, which we assume concave and increasing in R_i[2]. The systems are selfish in the sense that they only try to maximize their own utility. The rationality assumption means that each system will never choose a strictly dominated strategy[3]. We analyze the set of achievable rates in this non-cooperative scenario using non-cooperative game theory.

Once the set of non-cooperative achievable rates is determined, the operating point is chosen by the protocol to achieve efficiency and fairness. We can think of this protocol as a widely known standard that the systems can choose to follow or as a set of spectrum sharing rules imposed by the regulation authority. In either case, a system knows the protocol, which specifies how to act in every possible situation, but is free to comply with it or not. However, each system knows that all other systems comply with the protocol[4]. This last remark is key in our game theoretic analysis.

A. Short interaction between systems: one shot game

We first consider a static game of complete and perfect information, usually known as the Gaussian Interference Game (GIG) [6]. The complete information assumption is justified in Section V where we show that we can incentivize the systems to measure and exchange their parameters truthfully.

The game has M players, the M systems. The strategy space $\mathcal{S}_i$ of system i is the set of power allocations $p_i(f)$, $f \in [0, W]$ that satisfy the power constraint (3). A strategy s_i for user i is the choice of power allocation $p_i(f)$. For a given strategy profile $(s_1, \ldots, s_M)$ the rate of user i is given

[2]Notice the difference between the utility $U_i(R_i)$ of each individual system vs. the global utility $U(R_1, \ldots, R_M)$ introduced in Section III.

[3]A strategy s'_i for player i is strictly dominated by strategy s_i if $U_i(s_1, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_M) < U_i(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_M))$ for each $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_M)$ that can be constructed from the other players' strategy spaces.

[4]This is a consequence of the rationality assumption and the choice of a protocol that operates in a Nash equilibrium of a game.

by (2). The players play simultaneously, and know the utility functions of all the other players ($N_0, \{c_{i,j}\}_{i,j}, \{P_i\}_{i=1}^M, W$ are common knowledge). A strategy profile $\{s_i^*\}_{i=1}^M$ is a Nash Equilibrium (N.E.) of the game if

$$R_i(s_1^*, \ldots, s_M^*) \geq R_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_M^*) \quad \text{for all } s_i \in \mathcal{S}_i,\ i = 1, \ldots, M. \tag{6}$$

A direct consequence of the flat-fading and white noise assumption is the following fact:

Fact 1: The set of frequency-flat allocations $p_i(f) = P_i/W, f \in [0, W]$ for $i = 1, \ldots, M$ is a Nash Equilibrium of the GIG.

This means that the best possible strategy for a given system is to spread its available power over the total bandwidth whenever all the interfering systems are spreading their signals. Fact 1 can be understood by noting that the best response of a system to a strategy profile of the other systems is to waterfill the available power over the noise+interference seen. When all the other systems use flat allocation, the waterfilling power allocation is flat, and it follows that flat allocations are best responses to each other.

If the players randomize their actions, the (mixed) strategy of each player is the choice of probability distribution used for the randomization. The utility that each user gets is the expected utility, averaged over the random choices of actions of all the players. Taking into account these changes in the definition of the strategies and the utilities, the concept of a mixed strategy Nash equilibrium can be defined exactly as before.

When studying the set of N.E., one needs to consider both pure and mixed strategies. However, in the case of the GIG it turns out that we need only consider pure strategies.

Theorem 3: The GIG can only have pure strategy Nash equilibria. That is, every mixed strategy N.E. of the game must consist of atomic distributions with a single atom, and therefore is a pure strategy N.E.

Proof: See Appendix D. ■

If the channel gains across systems are sufficiently small the full-spread N.E. is the only N.E. of the Gaussian game. The following theorem gives a sufficient condition for the uniqueness of the full-spread N.E.

Theorem 4: If $\sum_{\substack{j=1 \\ j \neq i}}^M \frac{c_{j,i}}{c_{i,i}} < 1$ for $i = 1, \ldots, M$ then the full-spread N.E. is the only N.E. of the GIG.

Proof: See Appendix E. ■

Theorem 4 does not give us any information about the uniqueness of the Nash equilibrium when the condition $\sum_{\substack{j=1 \\ j \neq i}}^M \frac{c_{j,i}}{c_{i,i}} < 1$ for all i is not met.

While studying the equilibria of the iterative waterfilling algorithm, Luo and Pang derived independently in [19] more general sufficient conditions for the uniqueness of the Nash equilibrium than the one given in Theorem 4. Our condition is derived using a different method and provides additional insight into the problem.

In many cases, the set of rates that results from the full-spread N.E. is not Pareto efficient (i.e. is not in $\mathcal{R}^*$) so there may be a significant performance loss if the M systems operate in this point due to lack of cooperation. And in many cases this inefficient outcome is the only possible outcome of the game.

Consider for example a two system scenario (call it (d)) with $c_{1,1} = c_{2,2} = 1$, $c_{2,1} = c_{1,2} = 1/4$, $W = 1$, $N_0 = 1$ and $P_1 = P_2 = P$. Note that in this case the condition of Theorem 4 is satisfied. If both users spread their signals, they obtain rates $R_1^{FS} = R_2^{FS} = \log[1 + P/(1 + P/4)]$ [bits/s/Hz], which tend to $\log(5)$ [bits/s/Hz] as $P \to \infty$. However, if the systems orthogonalize their power allocations using half of the bandwidth each, the resulting rates are $R_1 = R_2 = (1/2)\log(1 + 2P)$ [bits/s/Hz], which tends to ∞ as $P \to \infty$. The regime in which $P \gg N_0$ corresponds to the high SNR regime. In this regime, when the systems orthogonalize their power allocations they can communicate with an interference free channel, and achieve large data rates. If on the contrary both systems spread their signals, the signal to interference plus noise ratio becomes limited by interference, resulting in a reduced communication rate. This example shows that the inefficiency resulting from choosing the full-spread equilibrium can be arbitrarily large.

B. Long term interactions: a repeated game

Scenario (d) shows that there are situations in which the only possible outcome of the game is very inefficient, and as a result, there is a large performance degradation due to lack of cooperation. This negative result can be attributed to the static nature of the game that we defined.

Many wireless systems operate and co-exist with the same set of competing systems over a long period of time. In this context, it may be more reasonable to model the scenario as a repeated (or dynamic) game where systems play multiple rounds, remembering the past experience in the choice of the power allocation in the next round. We will consider an infinite horizon repeated game, where the GIG is repeated forever. The utility of each player is defined by

$$U_i = (1 - \delta) \sum_{t=0}^{\infty} \delta^t R_i(t) \tag{7}$$

where $R_i(t)$ is the utility of user i in the stage game at time t, and $\delta \in (0, 1)$ is a discount factor that accounts for the delay sensitivity of the systems. At the end of each stage, all the players can observe the outcome of the stage-game and can use the complete history of play to decide on the future action. A strategy in the repeated game is a complete plan of action, that defines what the player will do in every possible contingency in which he may need to act.

One property of this repeated game is that sequences of strategy profiles that form a N.E. in the stage game, form a N.E. in the dynamic game[5]. Furthermore, the dynamic game allows for a much richer set of N.E. This is an advantage from the point of view of policy making or standardization. The systems can agree through a standardization process to operate in any N.E. of the dynamic game. Having many equilibrium points to choose from gives more flexibility in obtaining a fair and efficient resource allocation. A natural question that arises is what set of rates can be supported as a N.E. of the repeated game. The following theorem, a general version of which is

[5]For the reader familiar with game theory, these equilibria are in fact subgame perfect Nash equilibria.

due to Friedman [7], [8], gives a sufficient condition for the rate vector $(R_1,\dots,R_M)$ to be achievable as the resulting utilities in a N.E. of the repeated game.

Theorem 5: Let R_i^{FS} be the rate of system i when all the systems spread their power over the bandwidth W, i.e. the rate obtained in the full-spread N.E. There exists a sub-game perfect N.E.[6] of the dynamic GIG with utilities $(U_1,\dots,U_M)=(R_1,\dots,R_M)$ whenever $(R_1,\dots,R_M)\in\mathcal{R}$ and $R_i > R_i^{FS}$ for $i=1,\dots,M$ for a discount factor δ sufficiently close to 1.

Proof: Theorem C of [9] states that any utility vector that Pareto dominates the payoffs of a Nash equilibrium of the stage game can be supported by a sub-game perfect N.E. of the repeated game for a discount factor δ sufficiently close to 1. This Folk theorem is due to Friedman [7], [8], although he considered only Nash equilibria instead of perfect equilibria in his work. In the GIG the full-spread allocations form a N.E. (see Fact 1), and we can use $(R_1^{FS},\dots,R_M^{FS})$ as the payoff vector of the N.E. of the stage game in the Theorem above. ∎

Let $\{p_i(f)\}_{i=1}^M$ be the power allocations that result in the rate vector $(R_1,\dots,R_M)$ (which always exist since $(R_1,\dots,R_M)\in\mathcal{R}$). The strategy that each system follows to obtain the rate vector $(R_1,\dots,R_M)$ in Theorem 5 is the following trigger strategy:

- at $t=1$: use power allocation $p_i(f)$.
- at $t=t_0$: if at time $t=t_0-1$ every user $j\in\{1,\dots,M\}$ used the power allocation $p_j(f)$ then use $p_i(f)$. Otherwise spread the power over the total band, i.e. use the power allocation P_i/W for $f\in[0,W]$

The idea behind this strategy, is to "cooperate" by using the required power allocation as long as all the other systems cooperated in the previous stages. As soon as at least one system deviates from the "good" behavior, a punishment is triggered where all the other systems spread their powers forever. Since the rates obtained by the systems once the punishment is triggered are lower than those obtained with cooperation, it is in the system's own interest to cooperate. Friedman's analysis shows that if δ is not too small, the above set of strategies forms a sub-game perfect N.E.. The sub-game perfection property of the N.E. guarantees that each system will indeed apply the punishment once the punishing situation arises. This property makes the threats believable.

Applying these ideas to scenario (d), we can define a trigger strategy where system 1 uses the first half of the bandwidth, and system 2 uses the second half, as long as in all the previous stages both systems complied with this frequency allocation. If at some stage any of the systems stops complying, a punishment is triggered where the systems spread their powers forever. For large enough P this pair of strategies forms a N.E. where each system obtains a utility $1/2\log(1+2P)$. This shows how the punishment strategies within the dynamic game formulation allow us to overcome the inefficiency that we observed in the static game.

Theorem 5 gives us a sufficient condition for a rate vector $(R_1,\dots,R_M)$ to be achievable through a N.E. But if the condition of the theorem is not met we may still have hope to find some other N.E. to support the desired set of rates. A natural question to ask is if there are other N.E. that result in utilities $(R_1,\dots,R_M)$ with some $R_i < R_i^{FS}$. The following theorem answers this question negatively and provides a converse to Theorem 5.

Theorem 6: The rate R_i^{FS} is the reservation utility of player i in the GIG. That is, player i can obtain a utility at least as large as R_i^{FS} by using the power allocation $p_i(f)=P_i/W$, $f\in[0,W]$ regardless of the power allocations used by the other players. Therefore, the rate R_i obtained by user i in any N.E. of the GIG must satisfy $R_i\geq R_i^{FS}$. The same statement holds for the repeated GIG.

Proof: We will first prove that for a white Gaussian input, the worst possible Gaussian interference of given power is white. Since the power of each system is bounded to P_j, the total interference power seen by system i is bounded to $\sum_{j\neq i} c_{j,i}P_j$. We will prove that $I_i^*(f)=N_0+\sum_{j\neq i} c_{j,i}P_j/W$ for $f\in[0,W]$ minimizes

$$R_i=\int_0^W \log\left[1+\frac{c_{i,i}P_i}{WI_i(f)}\right]df$$

for noise+interference power bounded to $N_0W+\sum_{j\neq i} c_{j,i}P_j$.

First, we need only consider $I_i(f)$ satisfying $\int_0^W I_i(f)df = N_0W+\sum_{j\neq i} c_{j,i}P_j$, since increasing the interference power can only reduce R_i. We will consider only $I_i(f)$ that are continuous almost everywhere. Let f_1 and f_2 in $(0,W)$, be any continuity points of $I_i(f)$. Let $I=[I_i(f_1)+I_i(f_2)]/2$ and $\Delta=I_i(f_1)-I_i(f_2)$. Then for a small band δ around f_1 and f_2 the resulting rate of system i is:

$$\delta R_i=\delta\log\left[1+\frac{c_{i,i}P_i}{W(I+\Delta/2)}\right]+\delta\log\left[1+\frac{c_{i,i}P_i}{W(I-\Delta/2)}\right].$$

As was shown in the proof of Theorem 2 δR_i is minimized for $\Delta=0$. Therefore, we conclude that for the minimizing $I_i(f)$, $I_i(f_1)=I_i(f_2)$. Since f_1 and f_2 are arbitrary continuity points of $I_i(f)$, we have that the optimal $I_i(f)$ must be constant almost everywhere. It follows that $I_i(f)=I_i^*(f)$ almost everywhere.

If system i uses a white input, the worst case interference is obtained when all the other systems spread their powers, and it follows that a rate at least as large as R_i^{FS} is always achieved. Therefore, there is no incentive for player i to play any strategy that results in a utility smaller than R_i^{FS}. ∎

An immediate consequence of Theorems 5 and 6 is that if the desired operating point $(R_1,\dots,R_M)$ (i.e. the maximizer of a desired global utility) is component-wise greater than the spreading rate vector $(R_1^{FS},\dots,R_M^{FS})$ there is no performance loss due to lack of cooperation. However, when this condition is not satisfied, the best that one can do is to find the point $(R_1,\dots,R_M)\in\mathcal{R}^*$ that maximizes the global utility subject to $(R_1,\dots,R_M)\geq(R_1^{FS},\dots,R_M^{FS})$.

Referring to Figure 3 we see that in scenario (b) the optimal sum rate point lies within the achievable region in the non-cooperative setting. However, the optimal proportional fair point lies outside of this set and cannot be supported without cooperation. The best that one can do in the non-cooperative setting is to operate in the point indicated in the figure. In scenario (c) both the optimal sum rate and optimal

[6]The sub-game perfect N.E. is a refined and stronger version of the N.E. concept defined before. It guarantees that the N.E. does not arise due to unbelievable threats.

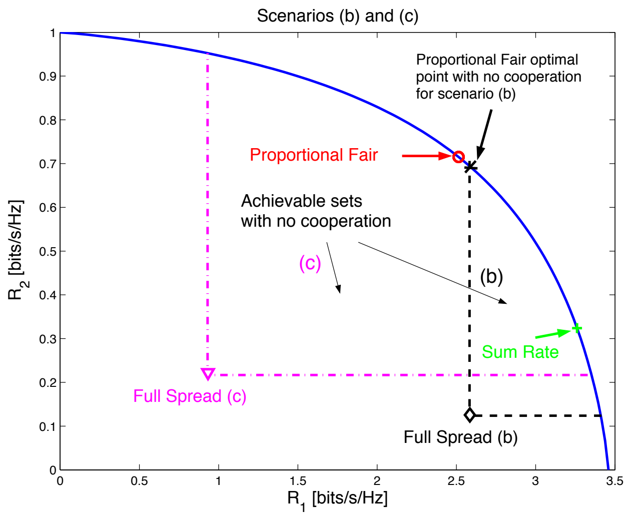

Fig. 3. Achievable rates with no cooperation for scenarios (b) and (c).

proportional fair rates are achievable in the non-cooperative setting. Note that while in the cooperative case the specific values of the cross gains had no influence on the achievable region (as long as the strong interference condition is satisfied) this is not true in the non-cooperative setting. This is because large cross gains enable the systems to apply punishments, and hence achieve a good N.E. through believable threats. In scenario (c) the large value of $c_{2,1}$ allows system 2 to punish system 1 whenever it departs from the proportional fair allocation.

To further illustrate the concepts introduced in this and the previous section, consider a two user scenario and assume that we use the proportional fair utility U_{PF} to measure the global performance. Without loss of generality we assume that $c_{1,1} = c_{2,2} = 1$, $W = 1$ and $N_0 = 1$. Also we take $P_1 = P_2 = P$ and analyze the results in terms of the $\mathsf{SNR} = P/N_0$. At a given SNR we can control the asymmetry between the two systems by varying the cross gains $c_{1,2}$ and $c_{2,1}$.

For a fixed set of parameters, using the results of Section III we optimize the power allocations to maximize the proportional fair metric, obtaining R_1^* and R_2^* as the resulting rates. In the non-cooperative scenario, R_1^* and R_2^* can only be supported by a N.E. if $R_1^* \geq R_1^{FS}$ and $R_2^* \geq R_2^{FS}$. If these inequalities are not satisfied, we obtain the best possible solution for the non-cooperative case by maximizing $\log(R_1) + \log(R_2)$ subject to the constraint $R_i \geq R_i^{FS}$, $i = 1, 2$, being $\tilde{R}_1$ and $\tilde{R}_2$ the corresponding optimal rates. If $R_i^* = \tilde{R}_i$ for $i = 1, 2$ we conclude that there is no loss due to lack of cooperation. If $R_i^* > \tilde{R}_i$ for $i = 1$ or $i = 2$ we measure the loss due to lack of cooperation using $\max_{i\in\{1,2\}} 100(R_i - \tilde{R}_i)/R_i$, i.e. the percentage loss in rate for one of the systems. Note that the other system will have a rate larger than the one obtained with cooperation. In Figure 4 we see that for low SNR, the region of rate pairs $(c_{1,2}, c_{2,1})$ for which there is a loss due to lack of cooperation is large, but this loss is quite small (about 5% in the worst case seen in the figure). As the SNR increases, this region becomes progressively smaller, but the corresponding performance loss is more significant. For $\mathsf{SNR} = 10$dB this performance loss

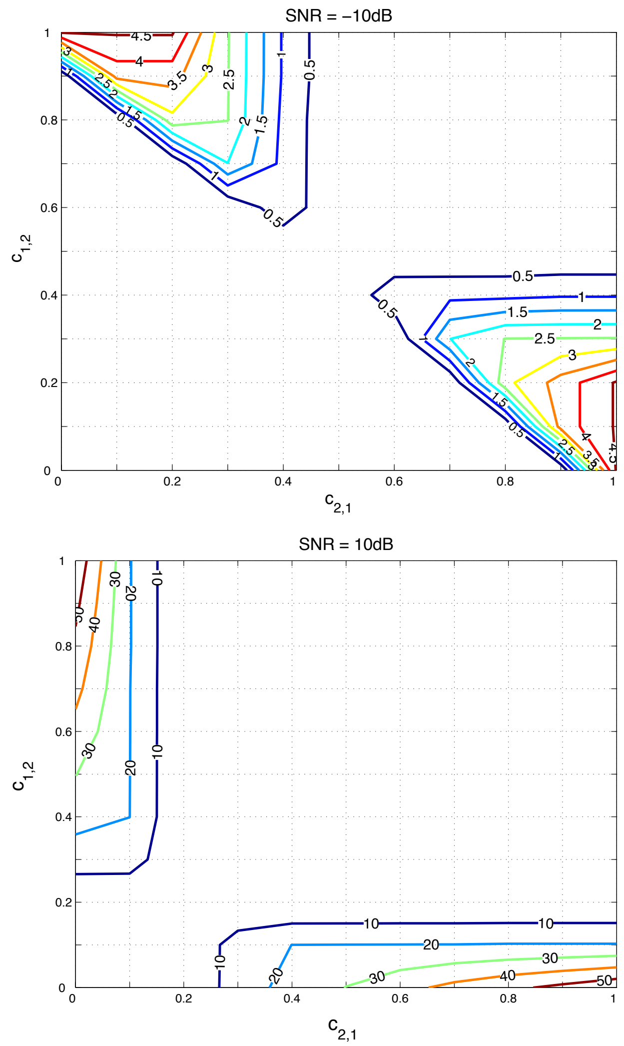

Fig. 4. Contour plots of $(R_i - \tilde{R}_i)/R_i(\%)$ as a function of the cross gains $c_{1,2}$ and $c_{2,1}$, for $\mathsf{SNR} = -10, 10$dB.

can be as large as 50% of the proportional fair rates, but this only occurs in very asymmetric situations.

At low SNR performance is limited by noise not interference, so whether the systems cooperate or not does not have much influence in performance. At larger SNRs, interference becomes the dominant performance limiter. In very asymmetric situations the full-spread point (R_1^{FS}, R_2^{FS}) is such that R_1^{FS} or R_2^{FS} is large. In either of these cases, one system obtains a large enough rate with the full-spread allocations, and a threat of the other system to apply the spreading punishment is not effective to modify its behavior. Using Theorem 6 we see that if R_1^{FS} or R_2^{FS} is large, the set of rates achievable in the non-cooperative situation is quite limited, and often it does not include the optimal cooperative point. Fortunately, the level of asymmetry required to reach this unfair situation increases with SNR. The same qualitative behavior should be observed with other performance metrics and larger number of users.

The plots of Figure 4 illustrate the performance loss due to lack of cooperation when there is asymmetry in the cross gains between the systems. One can do a similar analysis for the case when the source of asymmetry is the transmission power instead of the cross gains.

V. Parameter Measurement and Exchange between Selfish Systems

The games considered in Section IV were games of complete information. This means that all the parameters (power constraints, channel gains, etc.) are common knowledge to all the systems. In practice, some of these parameters have to be measured, and the corresponding measurements must be exchanged between the systems.

In a selfish environment it is possible that a system may try to tamper with the parameter measurement and exchange process in order to obtain an advantage. This situation can be modeled as a new game where we add in the action space of each system the different ways in which it can alter the parameter measurement and exchange process. In this section we show, under some assumptions, that this modified game has an efficient and fair Nash equilibrium for many reasonable fairness goals. We accomplish this by defining a parameter measurement and exchange protocol and showing that deviations from it can either be detected and punished or do not result in a better utility for the deviant system.

A. Channel measurement and exchange protocol

To measure the channel gains $\{c_{i,j}\}$ the transmitters of the different systems take turns in sending a pilot signal of normalized power. When transmitter i sends its pilot, all the receivers measure simultaneously the value of the channel gain $c_{i,j}$. After M time-slots all the channel gains are measured. We assume the existence of a low rate *control channel* that the systems can use to communicate with each other information about the parameters.

There are two ways in which a system may attempt to tamper with the measurement and exchange process: (a) a system may transmit the pilot with different power level than the nominal one; (b) a system may communicate fake channel measurement values. If these deviations pass undetected, the resource allocation algorithm may lead to solutions that are unfair and inefficient.

In order to detect such deviations we add some detection mechanisms to the protocol. These are:

- *Test messages*: System i transmits a test message using a predefined codebook of rate $W \log(1 + c_{i,j}P_i/N_0/W)$. If $c_{i,j}$ is the true value of the channel gain between transmitter i and receiver j, system j should be able to decode the message and feed it back to system i (using a low rate code) with negligible probability of error. If on the other hand system j has reported a value of $c_{i,j}$ larger than the true one, its error probability would be large, and the test would fail. Note that this test can only check whether the value of $c_{i,j}$ reported by system j is *larger* than the true value.
- *Multiple pilots*: All transmitters $j \neq i$ randomly select pilot powers P_j and transmit simultaneously white Gaussian signals. System j is required to broadcast the value of the total interference power received at its receiver. The other systems can then exchange the random power values and check whether the reported interference power matches $\sum_{j \neq i} c_{j,i} P_j$. Since system i does not know the values of P_j it can at most scale all values of $\{c_{j,i}\}_{j \neq i}$ by the same factor without being detected.
- *Triangulation*: This test uses a metric multi dimensional scaling technique to determine the location of transmitters and receivers from the channel gain measurements. The values of the channel gains cannot be arbitrary; they must correspond to actual locations of transmitters and receivers in a two or three dimensional space. In essence, the triangulation technique allows to detect inconsistencies between the measured channel gains and the path losses predicted by the path loss model using the estimated locations of transmitters and receivers. For the sake of brevity we omit the details of this technique, and limit the discussion to the kind of deviations from the protocol that can be detected.
  Triangulation can detect whether any system i scales all the channel gains $\{c_{i,j}\}$ (or $\{c_{j,i}\}$), $j \neq i$, by the same factor if either:
  a) the path loss model only takes into account the large-scale path loss (which is a function of the transmitter-receiver distance), or
  b) the path loss model includes also attenuation due to shadowing and fading, and these random attenuations are independent between different transmitter-receiver pairs.
  In case b) the detection accuracy improves as the number of transmitters and receivers increases, because one can use the increasing number of measurements to average out the randomness in the channel gains.
- *Rate detection*: Some system j decodes some of system i's messages, determines its communication rate, and feeds back the rate measurement to the other systems. This allows to detect if a system is communicating at a rate smaller or larger than the one assigned by the resource allocation algorithm. This technique requires that the signal to interference plus noise ratio (SINR) between the transmitter of system i and the receiver of system j be larger than the SINR of system i. In addition, system j needs to know the codebook used by system i, and should have the computational power to decode the message. Note that the probability of finding a system satisfying these requirements increases with the number of systems and the network density. In addition, note that if system j fails in decoding system i's message, the protocol could require some other system k, say, to decode. It follows that the technique is robust against decoding failures of specific systems which may be due to fading or other effects.

B. A new repeated game

We modify the repeated game definition of Section IV to add an initial stage of channel measurement, exchange, and verification. In this initial stage, each system can either comply with the protocol defined above or deviate from it. To deter selfish systems from deviating from the protocol we introduce punishment strategies as we did in Section IV. If in this initial stage any deviation is detected, a punishment is triggered in which all the systems spread their available power over the total bandwidth forever. In addition, if at any stage of the repeated game any system deviates from the power allocation or communication rate determined by the resource allocation algorithm, a similar punishment is triggered.

We want to show that the above punishment strategies form a Nash equilibrium in the modified repeated game whenever the corresponding game of Section IV had such an efficient and fair Nash equilibrium. Therefore we assume that the spreading rates obtained with punishment are smaller than the rates allocated by the resource allocation algorithm under the true channel gains.

The test messages, multiple pilots, and triangulation techniques (under the assumption of having a large and/or dense network) can detect any misrepresentation of the channel cross gains by a single deviant system. In addition, if a system reports a direct channel gain larger than the true value, the rate detection technique can detect that its communication rate is lower than the one required by the resource allocation algorithm, which assumes a direct gain larger than the real one.

Since any detected deviation from the protocol results in punishment, there is no incentive for any individual system to misrepresent channel cross gains, or to announce a direct gain larger than the true one.

It remains to show that there is no incentive for a system to report that its direct gain is smaller than its true value. Since the rate detection technique can measure the communication rate, if a system underestimates its direct gain, it will have to communicate at a rate compatible with the reported gain to avoid being detected as a deviant. Therefore, we need to compare R_i^*, the rate assigned to system i by the resource allocation algorithm when system i reports its true direct gain $c_{i,i}$ to the rate $\tilde{R}_i^*$ assigned to system i when it reports a direct gain $\tilde{c}_{i,i}$ with $\tilde{c}_{i,i} < c_{i,i}$.

One can show that if the global objective is to maximize the rate of the systems subject to a constant ratio among the rates (i.e. maximize R_1 subject to $R_i/R_1 = \alpha_i > 0$ for $i = 2, \ldots, M$) there is no incentive for any system to report a smaller value for its direct gain. In addition, if the strong interference condition $c_{i,j}c_{j,i} > c_{i,i}c_{j,j}$ holds for any pair of systems $i \neq j$, a similar result can be proved for the proportional fair ($U_{PF} = \sum_{i=1}^{M} \log(R_i)$), and weighted sum ($U_{WS} = \sum_{i=1}^{M} u_i R_i$ for $u_i > 0$) objectives as well[7].

In summary, we have shown that in a dense network under some assumptions on the fairness metrics and/or the channel gains, there is no performance loss due to selfish behavior even when the channel gains are not known a priori to the systems.

VI. Conclusions

Our game theoretic analysis showed that in many cases, the fair and efficient operating point can be enforced through punishment strategies. Moreover, the rates that can be obtained with our punishment strategies are essentially the best that one can hope for in a non-cooperative setting. Therefore our results are tight and quantify the best achievable performance with lack of cooperation. The two system example that we presented shows that in most situations the performance loss due to lack of cooperation is small, and vanishes with increasing SNR.

Finding efficient and fair spectrum allocations requires solving a complex optimization problem. We showed that under strong interference, frequency division multiplexing is optimal, and the resulting optimization problem is convex. In the more general case, the problem was shown to be of finite dimension but with non-convex structure. Finding efficient distributed algorithms that can be proved to converge to the optimal solution remains an open problem.

We assumed flat fading channels throughout the paper. An interesting problem for future research is to extend our results for a general frequency selective channel model. While the basic idea of enforcing cooperation through credible threats is applicable for the frequency selective case, the choice of punishments and methods to make them credible require further study.

The punishment strategies that we considered, which punish forever any deviation from the desired behavior, are optimal for obtaining the largest achievable rate region in a non-cooperative situation. However, in a practical setting where there may be measurement errors (in the parameters or in the observation of the result of previous stages of the game), punishing forever may be too strict. It is possible to modify the punishment to last a fixed number of stages, where this number is chosen long enough to deter any misbehavior.

Appendix A: Non-Gaussian Signals

The model introduced in Section II assumed that each system generated its codebook randomly using a Gaussian distribution. We now consider a more general model in which the different systems generate their codebooks using arbitrary distributions subject to a mean power constraint. We analyze whether Theorems 5 and 6 hold in this more general setting.

One can imagine that with more freedom in the choice of input distributions, it may be possible to exert a stronger punishment over a misbehaving system. In addition, a misbehaving system may want to find its most robust input signal to maximize its rate once the punishment is triggered. When system i misbehaves and the remaining systems apply a punishment, the interference observed at the receiver of system i has a maximum power given by $\sum_{j\neq i} c_{j,i}P_j$. It is possible to analyze the rates that system i gets for different input and punishment signals by considering a game between a sender x and an interferer z. Let $\mathbf{Y} = \mathbf{X} + \mathbf{Z}$, and assume that $\mathbf{X} \in \mathbb{C}^N$ and $\mathbf{Y} \in \mathbb{C}^N$ are independent, zero mean, circularly symmetric random vectors with covariance matrices $\mathbf{K_X}$ and $\mathbf{K_Z}$ subject to trace constraints $\text{tr}(\mathbf{K_X}) \leq P_X$ and $\text{tr}(\mathbf{K_Z}) \leq P_Z$. The strategy of each player is to choose a distribution subject to the power constraint. Define the payoff of player x to be $I(\mathbf{X};\mathbf{Y})$ and the payoff of player z to be $-I(\mathbf{X};\mathbf{Y})$. Since the payoffs sum to a constant (i.e. 0) the game is a zero-sum game. In our context $P_X = c_{i,i}P_i$ and $P_Z = \sum_{j\neq i} c_{j,i}P_j$.

The sender should choose its distribution $p_{\mathbf{X}}$ to solve $\sup_{p_{\mathbf{X}}} \inf_{p_{\mathbf{Z}}} I(\mathbf{X};\mathbf{Y})$ and the interferer should choose its

[7]The proofs of these results have been omitted for brevity

distribution $p_{\mathbf{Z}}$ to solve $\inf_{p_{\mathbf{Z}}} \sup_{p_{\mathbf{X}}} I(\mathbf{X};\mathbf{Y})$. In both cases, the supremum and infimum are taken over all distributions satisfying the power constraints P_X and P_Z respectively. In this way the sender maximizes its payoff under the worst possible interference, and the interferer maximizes its payoff when the sender can adapt its signal to the observed interference. It turns out that the game has a saddle point and a saddle value, that is, $\sup_{p_{\mathbf{X}}} \inf_{p_{\mathbf{Z}}} I(\mathbf{X};\mathbf{Y}) = \inf_{p_{\mathbf{Z}}} \sup_{p_{\mathbf{X}}} I(\mathbf{X};\mathbf{Y})$ which is achieved when both the input and interfering signals are white and Gaussian [11].

It follows that the strongest punishment that can be applied over a misbehaving system is achieved when all the other systems use white Gaussian signals. Therefore, the punishments used in Theorem 5 should still be white and Gaussian even if we have the freedom to choose other distributions. Moreover, by using a white Gaussian signal any system i can obtain a rate at least as large as R_i^{FS} regardless of the distribution of the interfering signals. Therefore Theorem 6 still holds under the more general model that we consider here. In particular, it is not possible to achieve any rate vector with a component $R_i < R_i^{FS}$ even allowing for arbitrary input distributions.

Appendix B: Proof of Theorem 1

Proof: We prove the theorem by defining a rate region $\tilde{\mathcal{R}}$ and showing that it can be achieved with piece-wise constant power allocations over at most $2M$ intervals. We then show that $\mathcal{R} \subseteq \tilde{\mathcal{R}}$.

Define the power vector $\mathbf{p} = (p(1), \ldots, p(M)) \in \mathbb{R}^M$ and the rate vector $\mathbf{r} = (r(1), \ldots, r(M)) \in \mathbb{R}^M$ where each rate is computed assuming frequency flat power allocations over the bandwidth W, i.e. $r(i) = W \log\left(1 + \frac{c_{i,i}p(i)}{N_0 W + \sum_{j\neq i} c_{j,i}p(j)}\right)$. Also define $\mathcal{A} \subset \mathbb{R}^{2M}$ by $\mathcal{A} = \{(\mathbf{r},\mathbf{p}) : \mathbf{p} \geq 0\}$.

We will first show that if $\mathcal{B} =$ Convex Hull $\{\mathcal{A}\}$ any point in $\mathcal{B}$ can be achieved with a convex combination of at most $2M$ points of $\mathcal{A}$. This follows directly from Carathéodory's theorem and it's extension (see [4], Theorem 18) by noting that $\mathcal{A}$ lies in $\mathbb{R}^{2M}$ and it is connected.

We then define the rate region

$$\tilde{\mathcal{R}} = \{\mathbf{r} : (\mathbf{r},\mathbf{p}) \in \mathcal{B} \text{ and } \mathbf{p} \leq (P_1, \ldots, P_M)\}$$

and show that $\mathcal{R} \subseteq \tilde{\mathcal{R}}$. Note that there is a one to one correspondence between points in $\tilde{\mathcal{R}}$ and piece-wise constant power allocations that satisfy the power constraints and are defined over at most $2M$ intervals.

We need to show that $\mathbf{r} \in \mathcal{R} \Rightarrow \mathbf{r} \in \tilde{\mathcal{R}}$. We will do this by first showing that if $\mathbf{r} \in \mathcal{R}$ then there is a sequence of rates $\{\mathbf{r}_n\}_n$ with $\mathbf{r}_n \in \tilde{\mathcal{R}}$ such that $\lim_{n\to\infty} \mathbf{r}_n = \mathbf{r}$. Finally, by showing that $\tilde{\mathcal{R}}$ is compact we will show that $\mathbf{r} \in \tilde{\mathcal{R}}$.

$\mathbf{r} \in \mathcal{R}$ implies that there exist power allocations $\{p_i(f)\}_{i=1}^M$ that satisfy the power constraints and that result in the rate vector $\mathbf{r} = (R_1, \ldots, R_M)$. Fix n, and partition the interval $[0, W)$ into n intervals $\{\mathcal{W}_k\}_{k=1}^n$ defined by $\mathcal{W}_k = [(k-1)W/n, k \cdot W/n)$. Let $\lambda_k = 1/n$ for $k = 1, \ldots, n$. Let $\tilde{p}_k(i) = W \cdot \inf_{f\in\mathcal{W}_k} p_i(f)$ and $\tilde{r}_k(i) = W \log\left(1 + \frac{c_{i,i}\tilde{p}_k(i)}{N_0 W + \sum_{j\neq i} c_{j,i}\tilde{p}_k(j)}\right)$ for $i = 1, \ldots, M$ and $k = 1, \ldots, n$. Finally let $\mathbf{p}_n = \sum_{k=1}^n \lambda_k \tilde{\mathbf{p}}_k$ and $\mathbf{r}_n = \sum_{k=1}^n \lambda_k \tilde{\mathbf{r}}_k$. Since $\int_0^W p_i(f)df = \sum_{k=1}^n \int_{\mathcal{W}_k} p_i(f)df \geq \sum_{k=1}^n \int_{\mathcal{W}_k} \tilde{p}_k(i)/W df = \sum_{k=1}^n W/n \cdot \tilde{p}_k(i)/W = \sum_{k=1}^n \lambda_k \tilde{p}_k(i) = p_n(i)$ for $i = 1, \ldots, M$, it follows that $\mathbf{p}_n \leq (P_1, \ldots, P_M)$ and hence $\mathbf{r}_n \in \tilde{\mathcal{R}}$. Also $\mathbf{r}_n$ is an approximation to the integral that defines $\mathbf{r}$ so it converges to it as the number of intervals n in which $[0, W]$ is partitioned goes to ∞.

To show that $\tilde{\mathcal{R}}$ is compact, we need to show that it is closed and bounded. This part of the proof is technical and is omitted for brevity. ■

Appendix C: Proof of Theorem 2

Proof: We use Theorem 1 to restrict attention to piece-wise constant power allocations.

We will prove the theorem by contradiction. Assume that $p_i(f)$ and $p_j(f)$ are not orthogonal in some interval, $\mathcal{W}_k$ say. Then, $p_i(f) = p_i > 0$ and $p_j(f) = p_j > 0$ for $f \in \mathcal{W}_k$. Also let $p_r(f) = p_r$ for $f \in \mathcal{W}_k$, $r \neq i, j$. Finally let $I_i = N_0 + \sum_{r\neq i,j} c_{r,i}p_r$ ($I_j = N_0 + \sum_{r\neq i,j} c_{r,j}p_r$) be the noise plus interference PSD received by system i (j) without taking into account the interference received from system j (i). We will show that the rates R_i and R_j can be increased simultaneously without decreasing the rates R_r, $r \neq i, j$, thus showing that $p_i(f)$ and $p_j(f)$ cannot result in Pareto efficient rates.

We first split $\mathcal{W}_k$ into two equal intervals $\mathcal{W}_{k,1}$ and $\mathcal{W}_{k,2}$. We then modify $p_i(f)$ in $\mathcal{W}_k$ as follows: $p_i(f) = p_i + \Delta_{i,1}/2$ for $f \in \mathcal{W}_{k,1}$ and $p_i(f) = p_i - \Delta_{i,1}/2$ for $f \in \mathcal{W}_{k,2}$. Note that the difference in power levels between the two intervals is $\Delta_{i,1}$, which we choose to be sufficiently small. We now run the iterative waterfilling algorithm[8] restricted to the interval $\mathcal{W}_k$ with power constraints $p_i|\mathcal{W}_k|$ and $p_j|\mathcal{W}_k|$, starting with the power allocation $p_j(f)$. When system j waterfills first, it observes a noise+interference with a difference in power levels between $\mathcal{W}_{k,1}$ and $\mathcal{W}_{k,2}$ of $c_{i,j}\Delta_{i,1}$. After waterfilling, its own power allocation $p_j(f)$ has a difference $\Delta_{j,1} = -\frac{c_{i,j}}{c_{j,j}}\Delta_{i,1}$ between the two sub-intervals. When system i waterfills, it observes a noise plus interference with difference $c_{j,i}\Delta_{j,1} = -\frac{c_{j,i}c_{i,j}}{c_{j,j}}\Delta_{i,1}$ and after waterfilling, its power allocation has a difference in power levels $\Delta_{i,2} = \frac{c_{i,j}c_{j,i}}{c_{i,i}c_{j,j}}\Delta_{i,1}$. Since by assumption $\frac{c_{i,j}c_{j,i}}{c_{i,i}c_{j,j}} > 1$, the power differences $\Delta_{i,n}$ and $\Delta_{j,n}$ increase in magnitude for increasing n, until the algorithm converges. Once the algorithm converges the resulting power allocations in the interval $\mathcal{W}_k$ satisfy the properties:

- The power differences have positive magnitude: $|\Delta_{i,\infty}| > 0$ and $|\Delta_{j,\infty}| > 0$.
- The power allocations are waterfilling solutions to each other.

We will now show that these allocations, restricted to the interval $\mathcal{W}_k$, result in higher rates for all systems than the initial flat allocations. Let $R_i(\Delta_j)$ be the rate associated to the interval $\mathcal{W}_k$ when system i uses a flat power allocation and system j uses a power allocation with a difference Δ_j between the intervals $\mathcal{W}_{k,1}$ and $\mathcal{W}_{k,2}$.

$$\begin{aligned} R_i(\Delta_j) &= \frac{|\mathcal{W}_k|}{2} \log\left[1 + \frac{c_{i,i}p_i}{I_i + c_{j,i}(p_j - \Delta_j/2)}\right] \\ &+ \frac{|\mathcal{W}_k|}{2} \log\left[1 + \frac{c_{i,i}p_i}{I_i + c_{j,i}(p_j + \Delta_j/2)}\right]. \end{aligned}$$

[8]The iterative waterfilling algorithm consists of letting each system distribute its available power over the noise+interference seen as if it was pouring liquid over a container. See [6] for a detailed explanation.

Differentiating $R_i(\Delta_j)$ with respect to Δ_j it is easy to check that $R_i'(\Delta_j) > 0$ for $\Delta_j > 0$ and $R_i'(\Delta_j) < 0$ for $\Delta_j < 0$. Therefore $R_i(\Delta_j)$ is minimized for $\Delta_j = 0$. Equivalently, for fixed total noise plus interference power, assuming that system i uses a flat power allocation, its rate is minimized when the total noise plus interference PSD is white. In addition, if system i waterfills over the noise plus interference seen, its rate can only increase with respect to the one obtained with a flat power allocation. So we conclude that after the iterative waterfilling algorithm converges, the rate of system i is larger than the initial rate obtained with flat allocations for systems i and j. A similar conclusion can be made for the rate of system j after convergence of iterative waterfilling.

In addition, the rates of the other systems do not decrease with the modified power allocations of systems i and j. If for example system r had zero power in the band k then modifying the power allocations of systems i and j does not change its rate. If on the other hand system r had positive power in the band, the result of modifying the power allocations of systems i and j using iterative waterfilling is to change the total noise plus interference seen by system r from a white PSD to a colored PSD of equal power. By the analysis above, this change in the PSD of the noise plus interference can only increase R_r. ■

APPENDIX D: PROOF OF THEOREM 3

Proof: In the Nash equilibrium, let F_P be the distribution used by system i to select the power allocation $p_i(f)$, and let F_I be the distribution function of the total noise plus interference seen by system i. The utility of system i at the N.E. is given by:

$$\begin{aligned} E[U_i] &= \int\int U_i\left\{\int_0^W \log\left[1+\frac{c_{i,i}p_i(f)}{I(f)}\right]df\right\}dF_P\ dF_I \\ &\underset{(a)}{\leq} \int U_i\left\{\int\int_0^W \log\left[1+\frac{c_{i,i}p_i(f)}{I(f)}\right]df\ dF_P\right\}dF_I \\ &\underset{(b)}{\leq} \int U_i\left\{\int_0^W \log\left[1+\frac{c_{i,i}\int p_i(f)dF_P}{I(f)}\right]df\right\}dF_I \end{aligned}$$

where (a) follows from the concavity of $U_i(\cdot)$, and (b) follows from the concavity of the $\log(\cdot)$ function, using Jensen's inequality in both cases. Since $\log(\cdot)$ is strictly concave, we can only have equality throughout when F_P corresponds to a constant random variable. Furthermore, $\int dF_P p_i(f)$ satisfies the power constraint P_i and hence system i can increase its utility by using the power allocation $\int dF_P p_i(f)$ whenever F_P is not atomic with a single atom. It follows that for F_P to be part of a N.E. it must correspond to a constant. ■

APPENDIX E: PROOF OF THEOREM 4

Proof: Let $\mathcal{P}$ be the set of vectors of power allocations $(\{p_1(f)\},\ldots,\{p_M(f)\})$ that satisfy the power constraints $P_1,\ldots,P_M$. The proof consists of showing that there exists a mapping $T:\mathcal{P}\rightarrow\mathcal{P}$ with two properties:

- The set of fixed points of T coincides with the set of Nash equilibria of the Gaussian Interference Game (GIG).
- T is a pseudocontraction with respect to some norm.

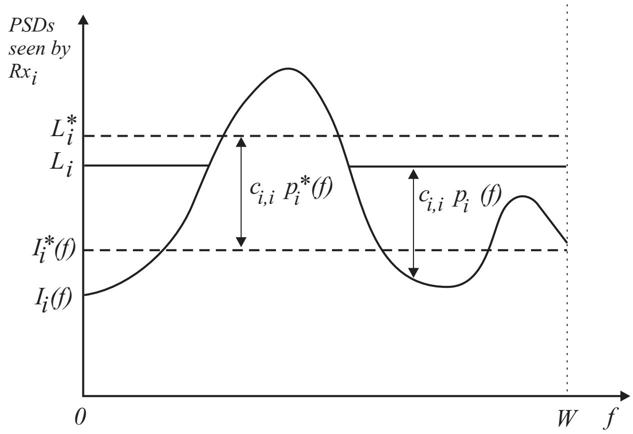

Fig. 5. Power spectral densities seen at the receiver of system i when it waterfills over the noise+interference $I_i(f)$ and $I_i^*(f)$.

Let $\mathbf{p}^*\in\mathcal{P}$ be a fixed point of T. T is a pseudocontraction with respect to some norm $\|\cdot\|$ if there exists $\alpha\in[0,1)$ such that:

$$\|T(\mathbf{p})-\mathbf{p}^*\|\leq\alpha\|\mathbf{p}-\mathbf{p}^*\| \quad (8)$$

for all $\mathbf{p}\in\mathcal{P}$. If $\tilde{\mathbf{p}}^*$ is any fixed point of T, the condition (8) implies that $\tilde{\mathbf{p}}^*=\mathbf{p}^*$, and hence T has a unique fixed point.

Given a power allocation vector $\mathbf{p} = (\{p_1(f)\},\ldots,\{p_M(f)\})$ we define $T(\mathbf{p})$ as the power allocation vector that results after each system waterfills its available power over the noise plus interference seen from the other systems, when they use the power allocations of $\mathbf{p}$. In other words, the ith component of $T(\mathbf{p})$ is the power allocation of system i when it waterfills its total power P_i over the noise plus interference observed from the other systems, when they use power allocations $p_1(f),\ldots,p_{i-1}(f),p_{i+1}(f),p_M(f)$. Since waterfilling is the best response of a system to a set of power allocations of the other systems, $\mathbf{p}$ is a fixed point of T iff $\mathbf{p}$ is a N.E. of the GIG.

We define the norm $\|\cdot\|$ as the maximum component norm:

$$\|\mathbf{p}\| = \max_{i\in\{1,\ldots,M\}}\sup_{f\in[0,W]}|p_i(f)|.$$

We make the technical assumption that the power allocations $\{p_i(f)\}$ are bounded, so that $\|\mathbf{p}\|<\infty$.

From Fact 1 we have that the set of flat allocations is a N.E. of the GIG, and hence is a fixed point $\mathbf{p}^*$ of T. We will now verify that T satisfies (8) with a modulus $\alpha=\max_i\sum_{j\neq i}\frac{c_{j,i}}{c_{i,i}}$ which, by assumption, is smaller than 1.

Let $\mathbf{p}\in\mathcal{P}$ and $\Delta=\|\mathbf{p}-\mathbf{p}^*\|$. Then $|p_i(f)-p_i^*(f)|\leq\Delta$ for $f\in[0,W]$ and $i=1,\ldots,M$. Let $I_i(f)$ $(I_i^*(f))$ be the noise+interference seen by system i when the other systems use power allocations from $\mathbf{p}$ $(\mathbf{p}^*)$. Then it follows that $|I_i(f)-I_i^*(f)|\leq\Delta\sum_{j\neq i}c_{j,i}$ for $f\in[0,W]$. Let $\tilde{p}_i(f)$ be the power allocation of system i after waterfilling over $I_i(f)$. There are two cases to consider:

1) $\tilde{p}_i(f)>0$ for all $f\in[0,W]$. In this case the "fluid" of system i covers all frequencies, and hence the total fluid level L_i is exactly the same as the one obtained when waterfilling over the noise plus interference profile $I_i^*(f)$. In this case we have $\tilde{p}_i(f)-p_i^*(f)=$

$(I_i^*(f) - I_i(f))/c_{i,i}$ and it follows that $|\tilde{p}_i(f) - p_i^*(f)| \leq \Delta \sum_{j \neq i} c_{j,i}/c_{i,i}$ for all $f \in [0, W]$, $i = 1, \ldots, M$. It follows that $\|T(\mathbf{p}) - \mathbf{p}^*\| \leq \alpha \|\mathbf{p} - \mathbf{p}^*\|$.

2) $\tilde{p}_i(f) = 0$ in some frequencies. In this case, the "fluid" of system i does not cover all frequencies, and hence the total fluid level L_i is lower than L_i^*, the level obtained when waterfilling over the noise plus interference profile $I_i^*(f)$ (See Figure 5). In order to bound $|\tilde{p}_i(f) - p_i^*(f)|$ we look at the lowest valley and highest peak of $\tilde{p}_i(f)$. If the total fluid level L_i was equal to L_i^*, the analysis of 1) would imply that at the highest peak, $\tilde{p}_i(f) - p_i^*(f) < \Delta \sum_{j \neq i} c_{j,i}/c_{i,i}$. But since $L_i < L_i^*$, this difference is even smaller. On the other hand, $\tilde{p}_i(f_1) = 0$ at some f_1 implies that at the frequency f where $I_i(f)$ reaches its highest peak (i.e. $f \in \arg\max_{\tilde{f}} I_i(\tilde{f})$), $(I_i(f) - I_i^*(f))/c_{i,i} > p_i^*(f)$, and hence $\Delta \sum_{j \neq i} c_{j,i}/c_{i,i} > p_i^*(f)$. Therefore, since at the frequency f we have $\tilde{p}_i(f) = 0$, it follows that $p_i^*(f) - \tilde{p}_i(f) < \alpha\Delta$. The same condition holds for any frequency f where $\tilde{p}_i(f) = 0$ (lowest valleys of $\tilde{p}_i(f)$), since $p_i^*(f)$ is constant. Therefore, it follows that for all frequencies $|\tilde{p}_i(f) - p_i^*(f)| \leq \alpha\Delta$ for $i = 1, \ldots, M$ and we conclude that $\|T(\mathbf{p}) - \mathbf{p}^*\| \leq \alpha \|\mathbf{p} - \mathbf{p}^*\|$.

■

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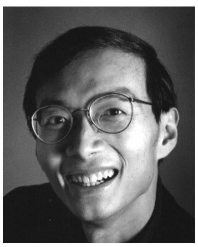

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