Statistical Tests for Presence of Cyclostationarity

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***Abstract*—Presence of kth-order cyclostationarity is defined in terms of nonvanishing cyclic-cumulants or polyspectra. Relying upon the asymptotic normality and consistency of kth-order cyclic statistics, asymptotically optimal χ^2 tests are developed to detect presence of cycles in the kth-order cyclic cumulants or polyspectra, without assuming any specific distribution on the data. Constant false alarm rate tests are derived in both time- and frequency-domain and yield consistent estimates of possible cycles present in the kth-order cyclic statistics. Explicit algorithms for $k \leq 4$ are discussed. Existing approaches are rather empirical and deal only with $k \leq 2$ case. Simulation results are presented to confirm the performance of the given tests.**

I. Introduction

THE concept of (almost) cyclostationarity and (almost) periodically time-varying ensemble statistics [14], [18], [21], [22] has gained a lot of interest in the engineering literature lately. A discrete-time zero-mean (almost) cyclostationary process, $x(t)$, is characterized by the property that its time-varying covariance $c_{2x}(t;\tau) = E\{x(t)x(t+\tau)\}$ accepts a Fourier series (FS) with respect to time t, as

$$c_{2x}(t;\tau) = \sum_{\alpha \in \mathcal{A}_2} C_{2x}(\alpha;\tau)e^{j\alpha t},$$

$$C_{2x}(\alpha;\tau) = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} c_{2x}(t;\tau)e^{-j\alpha t} \tag{1}$$

where the Fourier coefficient $C_{2x}(\alpha;\tau)$ is called the cyclic-covariance at cycle-frequency α and

$$\mathcal{A}_2 \triangleq \{\alpha : 0 \leq \alpha < 2\pi \text{ and } C_{2x}(\alpha;\tau) \neq 0\}. \tag{2}$$

Further, its time-varying spectrum $S_{2x}(t;\omega) \triangleq \sum_{\tau=-\infty}^{\infty} c_{2x}(t;\tau)\, e^{-j\omega\tau}$ can also be similarly expressed as

$$S_{2x}(t;\omega) = \sum_{\alpha \in \mathcal{A}_2} \mathcal{S}_{2x}(\alpha;\omega)e^{j\alpha t} \tag{3}$$

where the Fourier coefficient $\mathcal{S}_{2x}(\alpha;\omega) \triangleq \sum_\tau C_{2x}(\alpha;\tau)e^{-j\omega\tau}$ is called the cyclic-spectrum.

Cyclic-statistics have been used as tools for exploiting cyclostationarity in several applications, including communications [7], [8], signal processing [4], [5], [10]–[13], [16], [27], [29], hydrology [30], multivariate analysis [25], and array processing [32], for developing improved SNR, parametric and nonparametric algorithms in nonstationary environments.

An important assumption made in these algorithms is that the cycles present in the signal statistics of interest (e.g., the set $\mathcal{A}_2$ of (2)) are known. This knowledge makes estimation of the ensemble-statistics possible, and allows for separation in the cyclic-domain of signals with distinct cycles. However, in several applications, the cycle frequencies are not known *a priori*. For example, in processing of signals from a mechanically vibrating system one may not know the cycles arising due to unknown vibration modes. For other applications such as identification of periodically time-varying systems [4], [11], [29] or processing of signals with periodically missing observations [4], [5], [12], the knowledge of cycles is necessary to determine the unknown period.

An attempt was made in [33] to define a "degree of cyclostationarity." However, no statistical test was provided to check for the presence of cycles. An interesting graphical method for presence of second-order cyclostationarity was developed in [19] by modifying Goodman's test for non-stationarity [15]. The resulting test is rather empirical and is not geared towards checking for presence of cycles, due to the nature of Goodman's test. Further, with the growing interest in higher than second-order cyclostationarity [4]–[6], [9], [11]–[13], [27], [29], there is also a need to detect presence of cycles in the kth-order cyclic-cumulants and polyspectra for $k \geq 3$.

In this paper, tests are developed to check for presence of cycles in the cyclic-covariance and spectrum by exhaustively searching over candidate cycles for which the corresponding cyclic-statistics are nonzero and statistically significant. Asymptotically χ^2, constant false alarm rate (CFAR) tests are derived in both time- and frequency-domain using the asymptotic normality of sample cyclic-covariance and spectrum, respectively, without requiring knowledge of the data distribution. Generalizations for detecting presence of cycles in the kth-order cyclic-cumulants and polyspectra are also developed. Apart from providing estimates of possible cycle frequencies, these tests inherently check for presence of cyclostationarity and are expected to be performed as a first step in most algorithms that exploit cyclostationarity. Differences and similarities between the proposed methods and those of [33] and [19] are delineated wherever appropriate.

To illustrate the elements of our approach, we first present the time-domain tests with $k = 2$, for simplicity, in Section II. The kth-order generalizations for $k \geq 1$ are derived in Section III, and frequency-domain tests are developed in Section IV. Simulation results are presented in Section V and, finally, conclusions are drawn in Section VI.

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II. Time-Domain Test

The objective of this section is to derive a test for finding the cycles present in $c_{2x}(t;\tau)$ of (1), for a fixed τ. In other words, we wish to detect and estimate those α's for which $\mathcal{C}_{2x}(\alpha;\tau) \neq 0$, from a given stretch of data $\{x(t)\}_{t=0}^{T-1}$. If there exists one pair $(\alpha;\tau)$ for which $\mathcal{C}_{2x}\ (\alpha;\tau) \neq 0$, then we say that second-order cyclostationarity is present in $x(t)$. Hence, by detecting and estimating cycles of $c_{2x}(t;\tau)$ we are essentially testing for the presence of cyclostationarity. For convenience, we study zero-mean processes although this assumption is not restrictive, as shown in Section II-B.

To check if $\mathcal{C}_{2x}(\alpha;\tau)$ in (1) is null for a given candidate cycle consider the following consistent estimator of $\mathcal{C}_{2x}(\alpha;\tau)$ (see Section II-A)

$$\hat{\mathcal{C}}_{2x}(\alpha;\tau) \triangleq \frac{1}{T}\sum_{t=0}^{T-1} x(t)x(t+\tau)e^{-j\alpha t} \tag{4}$$

$$= \mathcal{C}_{2x}(\alpha;\tau) + \epsilon_{2x}^{(T)}(\alpha;\tau) \tag{5}$$

where $\epsilon_{2x}^{(T)}(\alpha;\tau)$ represents the estimation error which vanishes asymptotically as $T \to \infty$. Due to the error $\epsilon_{2x}^{(T)}(\alpha;\tau)$, the estimator $\hat{\mathcal{C}}_{2x}^{(T)}(\alpha;\tau)$ is seldom exactly zero in practice, even if α is not a cycle frequency. This raises an important issue about deciding whether a given value of $\hat{\mathcal{C}}_{2x}^{(T)}(\alpha;\tau)$ is "zero" or not. To answer this question statistically, we formulate the decision-making problem in a generalized hypotheses-testing framework.

In general, we consider a vector of $\hat{\mathcal{C}}_{2x}^{(T)}(\alpha;\tau)$ rather than a single value in order to check simultaneously for the presence of cycles in a set of lags τ.

Let $\tau_1,\ldots,\tau_N$ be a fixed set of lags, α be a candidate cycle-frequency, and

$$\hat{\mathbf{c}}_{2x}^{(T)} \triangleq \Big[\mathrm{Re}\Big\{\hat{\mathcal{C}}_{2x}^{(T)}(\alpha;\tau_1)\Big\},\ldots,\mathrm{Re}\Big\{\hat{\mathcal{C}}_{2x}^{(T)}(\alpha;\tau_N)\Big\}, \mathrm{Im}\Big\{\hat{\mathcal{C}}_{2x}^{(T)}(\alpha;\tau_1)\Big\},\ldots,\mathrm{Im}\Big\{\hat{\mathcal{C}}_{2x}^{(T)}(\alpha;\tau_N)\Big\}\Big] \tag{6}$$

represent a $1 \times 2N$ *row* vector consisting of second-order cyclic-cumulant estimators from (4) with Re{ } and Im{ } representing the real and imaginary parts, respectively. If the asymptotic (true) value of $\hat{\mathbf{c}}_{2x}^{(T)}$ is given as $\mathbf{c}_{2x}$

$$\mathbf{c}_{2x} \triangleq [\mathrm{Re}\{\mathcal{C}_{2x}(\alpha;\tau_1)\},\ldots,\mathrm{Re}\{\mathcal{C}_{2x}(\alpha;\tau_N)\}, \mathrm{Im}\{\mathcal{C}_{2x}(\alpha;\tau_1)\},\ldots,\mathrm{Im}\{\mathcal{C}_{2x}(\alpha;\tau_N)\}] \tag{7}$$

then using (5), we can write

$$\hat{\mathbf{c}}_{2x}^{(T)} = \mathbf{c}_{2x} + \boldsymbol{\epsilon}_{2x}^{(T)} \tag{8}$$

where $\boldsymbol{\epsilon}_{2x}^{(T)} \triangleq [\mathrm{Re}\{\epsilon_{2x}^{(T)}(\alpha;\tau_1)\},\ldots,\mathrm{Re}\{\epsilon_{2x}^{(T)}(\alpha;\tau_N)\}, \mathrm{Im}\{\epsilon_{2x}^{(T)}(\alpha;\tau_1)\},\ldots,\mathrm{Im}\{\epsilon_{2x}^{(T)}(\alpha;\tau_N)\}]$, is the estimation error vector. To check if α is a cycle frequency or not we formulate the following hypothesis testing problem:

$$\mathbf{H}_0{:}\alpha \notin \mathcal{A}_2\ \forall\{\tau_n\}_{n=1}^{N} \Longrightarrow \hat{\mathbf{c}}_{2x}^{(T)} = \boldsymbol{\epsilon}_{2x}^{(T)}$$
$$\mathbf{H}_1{:}\alpha \in \mathcal{A}_2 \text{ for some } \{\tau_n\}_{n=1}^{N} \Longrightarrow \hat{\mathbf{c}}_{2x}^{(T)} = \mathbf{c}_{2x} + \boldsymbol{\epsilon}_{2x}^{(T)}. \tag{9}$$

Since $\mathbf{c}_{2x}$ is nonrandom, the distribution of $\hat{\mathbf{c}}_{2x}$ under $\mathbf{H}_0$ and $\mathbf{H}_1$ differs only in mean. Therefore, testing for the presence of a given α in $\mathcal{A}_2$ is equivalent to a binary classification problem and requires knowledge of the distribution of $\boldsymbol{\epsilon}_{2x}^{(T)}$ for designing a decision strategy. Because the distribution of the data is *unknown*, we subsequently make use of the asymptotic properties of the cyclic-covariance estimators to infer the asymptotic distribution of $\boldsymbol{\epsilon}_{2x}^{(T)}$.

A. Asymptotics of Cyclic-Covariance Estimators

The time-varying kth-order moment $m_{kx}(t;\boldsymbol{\tau}), \boldsymbol{\tau} \triangleq (\tau_1, \ldots, \tau_{k-1})$ of a process, $x(t)$, is defined as

$$m_{kx}(t;\boldsymbol{\tau}) = E\{x(t)x(t+\tau_1)\cdots x(t+\tau_{k-1})\}. \tag{10}$$

With $\tau_0 \triangleq 0$, let $m_\nu(t;\boldsymbol{\tau}_\nu) \triangleq E\{x(t+\tau_{\nu_1})\cdots x(t+\tau_{\nu_q})\}$, where $\{\boldsymbol{\tau}_\nu = [\tau_{\nu_1},\ldots,\tau_{\nu_q}]\}$ and ν denotes the group $\tau_{\nu_1},\ldots,\tau_{\nu_q}$. The kth-order cumulant, $c_{kx}(t;\boldsymbol{\tau})$, of $x(t)$, is defined as

$$c_{kx}(t;\boldsymbol{\tau}) = \sum_{\nu}(-1)^{p-1}(p-1)! \times m_{\nu_1 x}(t;\tau_{\nu_1})\cdots m_{\nu_p x}(t;\tau_{\nu_p}) \tag{11}$$

where the summation on ν extends over all partitions $\nu = \nu_1\cup\cdots\cup\nu_p$ of $\{\tau_0,\ldots,\tau_{k-1}\}$, with p representing the number of groups in a partition. The mixing condition that we require for deriving the asymptotic properties of $\hat{\mathcal{C}}_{2x}^{(T)}(\alpha;\tau)$ is given in terms of $c_{kx}(t;\boldsymbol{\tau})$ as follows:

$$\mathbf{A1}\ \sum_{\boldsymbol{\tau}=-\infty}^{\infty} \sup_t |\tau_l c_{kx}(t;\boldsymbol{\tau})| < \infty, l \in \{1,\ldots,k-1\}, \forall k.$$

Intuitively, assumption **A1** implies that samples of the process $x(t)$ that are well separated in time are approximately independent. **A1** is met, for example, by stable (non)stationary linear processes (see also [4]).

With these preliminaries we are ready to present the main result of this section, namely, the asymptotic properties of $\hat{\mathcal{C}}_{2x}^{(T)}(\alpha;\tau)$: Let $x(t)$ be a discrete-time generally complex valued cyclostationary process and define

$$f(t;\tau) \triangleq x(t)x(t+\tau). \tag{12}$$

Let the unconjugated and conjugated cyclic-spectrum of $f(t;\tau)$ be defined, respectively, as

$$\mathcal{S}_{2f_{\tau,\rho}}(\alpha;\omega) \triangleq \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\sum_{\xi=-\infty}^{\infty}\mathrm{cum}\{f(t;\tau), f(t+\xi;\rho)\}e^{-j\omega\xi}e^{-j\alpha t}$$

$$\mathcal{S}_{2f_{\tau,\rho}}^{(*)}(\alpha;\omega) \triangleq \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\sum_{\xi=-\infty}^{\infty}\mathrm{cum}\{f(t;\tau), f^*(t+\xi;\rho)\}e^{-j\omega\xi}e^{-j\alpha t}$$

where $*$ denotes complex conjugation, and $(*)$ is just notational. Notice that for real processes, $f(t;\tau)$ is real, and hence, $\mathcal{S}_{2f_{\tau,\rho}}(\alpha;\omega) \equiv \mathcal{S}_{2f_{\tau,\rho}}^{(*)}(\alpha;\omega)$.

Theorem 1: If $x(t)$ satisfies assumption **A1**, then the estimator $\hat{C}_{2x}^{(T)}(\alpha;\tau)$ defined in (4) is mean-square sense consistent; i.e.

$$\lim_{T\to\infty} \hat{C}_{2x}^{(T)}(\alpha;\tau) \overset{m.s.s}{=} C_{2x}(\alpha;\tau). \tag{13}$$

Additionally, $\sqrt{T}[\hat{C}_{2x}^{(T)}(\alpha;\tau) - C_{2x}(\alpha;\tau)]$ is asymptotically complex normal with covariances given by

$$\lim_{T\to\infty} T\, cum\left\{\hat{C}_{2x}^{(T)}(\alpha;\tau), \hat{C}_{2x}^{(T)}(\beta;\rho)\right\} = S_{2f_{\tau,\rho}}(\alpha+\beta;\beta)$$
$$\lim_{T\to\infty} T\, cum\left\{\hat{C}_{2x}^{(T)}(\alpha;\tau), \hat{C}_{2x}^{*(T)}(\beta;\rho)\right\} = S_{2f_{\tau,\rho}}^{(*)}(\alpha-\beta;-\beta). \tag{14}$$

Proof: See Appendix A. □

B. Test Statistics for $k = 2$

Returning to our test for presence of cycles, i.e., (9), we observe from the asymptotic normality of cyclic-cumulant estimators (c.f. Theorem 1 and (5)) that

$$\lim_{T\to\infty} \sqrt{T}\boldsymbol{\epsilon}_{2x}^{(T)} \overset{D}{=} \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{2c}) \tag{15}$$

where $\overset{D}{=}$ denotes convergence in distribution, $\mathcal{N}$ stands for a multivariate normal density, and $\boldsymbol{\Sigma}_{2c}$ is the asymptotic covariance matrix computed using (14) as follows Let $\mathbf{Q}_{2c}$ and $\mathbf{Q}_{2c}^{(*)}$ be two covariance matrices with (m,n)th entries given, respectively, as

$$\mathbf{Q}_{2c}(m,n) \triangleq S_{2f_{\tau_m,\tau_n}}(2\alpha;\alpha)$$
$$\mathbf{Q}_{2c}^{*}(m,n) \triangleq S_{2f_{\tau_m,\tau_n}}^{(*)}(0;-\alpha). \tag{16}$$

Using (14) and (16) and elementary complex algebraic manipulations it follows that the (auto- and cross-) covariance of the real and imaginary parts of $\hat{C}_{2x}^{(T)}(\alpha;\tau_n), 1 \le n \le N$ are computed as

$$\boldsymbol{\Sigma}_{2c} = \begin{bmatrix} \mathrm{Re}\left\{\frac{\mathbf{Q}_{2c}+\mathbf{Q}_{2c}^{(*)}}{2}\right\} & \mathrm{Im}\left\{\frac{\mathbf{Q}_{2c}-\mathbf{Q}_{2c}^{(*)}}{2}\right\} \\ \mathrm{Im}\left\{\frac{\mathbf{Q}_{2c}+\mathbf{Q}_{2c}^{(*)}}{2}\right\} & \mathrm{Re}\left\{\frac{\mathbf{Q}_{2c}^{(*)}-\mathbf{Q}_{2c}}{2}\right\} \end{bmatrix} \tag{17}$$

From (9) and (15), it is seen that the hypotheses-testing problem of (9) is asymptotically equivalent to the one of checking for nonzeroness of the unknown mean of a multivariate normal random variable. This problem is equivalent to a generalized maximum-likelihood detection problem in the cyclic-statistics domain and as usually done [17], [28], (pp. 378–380 of [31]), we let the following generalized likelihood function be our test statistic[1]

$$\mathcal{T}_{2c} \triangleq T\hat{\mathbf{c}}_{2x}^{(T)}\hat{\boldsymbol{\Sigma}}_{2c}^{-1}\hat{\mathbf{c}}_{2x}^{(T)'} \tag{18}$$

where $\hat{\boldsymbol{\Sigma}}_{2c}$ represents a consistent estimator of $\boldsymbol{\Sigma}_{2c}$ to be developed subsequently in Section IV. To set a threshold for hypotheses testing, we next derive the asymptotic distribution of $\mathcal{T}_{2c}$. The following theorem comes in handy (see pp. 337 and 344 of [24] for a proof):

[1] In (18), prime denotes transpose, and the pseudo-inverse must replace the inverse if $\hat{\Sigma}_{2c}$ is rank deficient.

Theorem 2: **R1)** If the $1 \times M$ vector $\hat{\theta}^{(T)}$ is an asymptotically Gaussian estimator of θ, based on T data samples, i.e., $\lim_{T\to\infty}\sqrt{T}(\hat{\theta}^{(T)} - \theta) \overset{D}{=} \mathcal{N}(\mathbf{0}, \Sigma_\theta)$ and $\theta \ne \mathbf{0}$, then $\lim_{T\to\infty}\sqrt{T}(\hat{\theta}^{(T)}\hat{\theta}^{(T)'} - \theta\theta') \overset{D}{=} \mathcal{N}(\mathbf{0}, 4\theta\Sigma_\theta\theta')$, but if $\theta \equiv \mathbf{0}$ and $\Sigma_\theta \equiv \mathbf{I}$ (an identity matrix), then $\lim_{T\to\infty} T(\hat{\theta}^{(T)}\hat{\theta}^{(T)'} - \theta\theta') \overset{D}{=} \chi_M^2$ where χ_M^2 denotes a central chi-squared distribution with M degrees of freedom.

R2) If $\lim_{T\to\infty}\hat{\theta}^{(T)} \overset{D}{=} \theta$ and $\lim_{T\to\infty}\hat{\Psi}^{(T)} \overset{m.s.s}{=} \Psi$, an $M\times M$ matrix, then $\lim_{T\to\infty}\hat{\theta}^{(T)}\hat{\Psi}^{(T)} \overset{D}{=} \theta\Psi$. □

Let $\hat{\boldsymbol{\theta}}^{(T)} \triangleq \hat{\mathbf{c}}_{2x}^{(T)}\hat{\boldsymbol{\Sigma}}_{2c}^{-1/2}$ and $M = 2N$. Since $\hat{\boldsymbol{\Sigma}}_{2c}^{-1/2}$ is mean-square sense consistent, and $\hat{\mathbf{c}}_{2x}^{(T)}$ is asymptotically normal (see Theorem 1) it follows from **R2**) of Theorem 2 that $\hat{\boldsymbol{\theta}}^{(T)}$ is asymptotically normal. Then, using (9) and **R2**) of Theorem 2, it follows that $\mathcal{T}_{2c}$ in (18) has the following asymptotic distribution under $\mathbf{H}_0$

$$\lim_{T\to\infty} \mathcal{T}_{2c} \overset{D}{=} \chi_{2N}^2 \tag{19}$$

and under the alternative hypothesis $\mathbf{H}_1$

$$\lim_{T\to\infty} \sqrt{T}\left(\hat{\mathbf{c}}_{2x}^{(T)}\hat{\boldsymbol{\Sigma}}_{2x}^{-1}\hat{\mathbf{c}}_{2x}^{(T)'} - \mathbf{c}_{2x}\boldsymbol{\Sigma}_{2x}^{-1}\mathbf{c}_{2x}'\right) \overset{D}{=} \mathcal{N}(0, 4\mathbf{c}_{2x}\boldsymbol{\Sigma}_{2x}^{-1}\mathbf{c}_{2x}'). \tag{20}$$

Using (19) and (20), we now present our test which is based on a constant false alarm rate approach for selecting a threshold. For a given probability of false-alarms, $P_F \triangleq \Pr\{\mathcal{T}_{2c} \ge \Gamma \mid \mathbf{H}_0\}$, we find a threshold Γ from the central χ^2 tables with $2N$ degrees of freedom [20], such that (see (19)) $P_F = \Pr\{\chi_{2N}^2 \ge \Gamma\}$. Our test is given as

$$\text{if } \mathcal{T}_{2c} \ge \Gamma \text{ declare } \alpha \in \mathcal{A}_2 \text{ for some } \tau_1,\ldots,\tau_N$$
$$\text{else declare } \alpha \notin \mathcal{A}_2, \forall \tau_1,\ldots,\tau_N. \tag{21}$$

Once the threshold Γ in (21) has been set, one can approximately evaluate the probability of detection, $P_D \triangleq \Pr\{\mathcal{T}_{2c} \ge \Gamma \mid \mathbf{H}_1\}$, using the distribution of $\mathcal{T}_{2c}$ under $\mathbf{H}_1$. From (20) and for T large enough, we may approximately write the distribution of $\mathcal{T}_{2c}$ as

$$\mathcal{T}_{2c} \sim \mathcal{N}(T\mathbf{c}_{2x}\boldsymbol{\Sigma}_{2c}^{-1}\mathbf{c}_{2x}', 4T\mathbf{c}_{2x}\boldsymbol{\Sigma}_{2c}^{-1}\mathbf{c}_{2x}'). \tag{22}$$

Therefore, P_D can be evaluated in practice by substituting for $\mathbf{c}_{2x}$ and $\boldsymbol{\Sigma}_{2c}$ in (22) by their estimates and using the standard normal tables [20]. The algorithm for $k = 2$ can be implemented using the following steps:

Step 1: From the available data and using (4), compute $\hat{\mathbf{c}}_{2x}^{(T)}$ as in (6).

Step 2: Fill in the entries of the covariance matrix $bf\Sigma_{2c}$ using the consistent cyclic-spectrum estimator of Section IV, specifically, with $F_{T,\tau}(\omega) = \sum_{t=0}^{T-1} x(t)x(t+\tau)e^{-j\omega t}$ compute

$$\hat{S}_{2f_{\tau_m,\tau_n}}^{(T)}(2\alpha;\alpha) = \frac{1}{TL}\sum_{s=-(L-1)/2}^{(L-1)/2} W^{(T)}(s) \times F_{T,\tau_n}\left(\alpha - \frac{2\pi s}{T}\right)F_{T,\tau_m}\left(\alpha + \frac{2\pi s}{T}\right)$$

$$\hat{\mathcal{S}}^{(*T)}_{2f_{\tau_m,\tau_n}}(0;-\alpha) = \frac{1}{TL}\sum_{s=-(L-1)/2}^{(L-1)/2} W^{(T)}(s) \times F^{*}_{T,\tau_n}\left(\alpha+\frac{2\pi s}{T}\right)F_{T,\tau_m}\left(\alpha+\frac{2\pi s}{T}\right) \tag{23}$$

where $W^{(T)}$ is a spectral window of length L (odd). Using (23), fill in the entries of $\hat{\mathbf{Q}}_{2c}$ and $\hat{\mathbf{Q}}^{(*)}_{2c}$ via (16), which in turn yields $\hat{\mathbf{\Sigma}}_{2c}$, as in (17).

Step 3: Compute the test statistic as $\mathcal{T}_{2c} = T\hat{\mathbf{c}}^{(T)}_{2x}\hat{\mathbf{\Sigma}}^{-1}_{2c}\hat{\mathbf{c}}^{(T)'}_{2x}$.

Step 4: For a given probability of false alarms, P_F, using central χ^2 tables for $2N$ degrees of freedom [20], find a threshold Γ so that $P_F = \Pr\{\chi^2 > \Gamma\}$.

Step 5: Declare α as a cycle frequency at least for one of $\tau_1,\ldots,\tau_N$, if $\mathcal{T}_{2c} \geq \Gamma$; else, decide that α is not a cycle frequency of $\mathcal{C}_{2x}(\alpha;\tau)$ for any of $\tau_1,\ldots,\tau_N$.

The test summarized in these steps exploits the asymptotic normality of cyclic statistics to introduce a χ^2 CFAR test, which is asymptotically optimal in the generalized likelihood sense (see pp. 378–380 of [31]). Further, our test is based on the cyclic-statistics only and does not require knowledge about the data distribution. The variance normalization in the test statistic makes the thresholding easier and standard by employing table look-up from standard central χ^2 tables, irrespective of the particular signal at hand. Also, the χ^2_{2N} distribution under the null hypothesis is asymptotically exact, therefore we guarantee that the observed false alarm rates converge to the one corresponding to the threshold selection. In this sense our test is consistent, which is also an aspect of its optimality.

By performing the given test for various values of α one can determine possible cycles in $\hat{\mathcal{C}}^{(T)}_{2x}(\alpha;\tau), \tau=\tau_1,\ldots,\tau_N$. Presence of cycles indicates presence of second-order cyclostationarity, however, to declare absence one has to show that $\hat{\mathcal{C}}^{(T)}_{2x}(\alpha;\tau)\equiv 0, \forall(\alpha;\tau)$, in the sense of the test. If there exists an $(\alpha;\tau)$ for which $\mathcal{C}_{2x}(\alpha;\tau)\neq 0$, then there is *presence* of second-order cyclostationarity. However, this does not imply that $x(t)$ is cyclostationary, as is seen from the following example.

Example 1: Let $x(t) = w_1(t)\cos(\omega_0 t) + b(t)w_2(t)$, where $w_1(t)$ and $w_2(t)$ are zero-mean uncorrelated stationary random processes and $b(t)$ is a deterministic transient function which is nonzero only for $0 \leq t \leq T_0$. Using the definition $c_{2x}(t;\tau) = E\{x(t)x(t+\tau)\}$ it follows from the FS in (1) that

$$c_{2x}(t;\tau) = \frac{c_{2w_1}(\tau)}{4}e^{-j\omega_0\tau}e^{-j2\omega_0 t} + \frac{c_{2w_1}(\tau)}{2} \times \cos\omega_0\tau + \frac{c_{2w_1}(\tau)}{4}e^{j\omega_0\tau}e^{j2\omega_0 t} + \bar{c}(t;\tau)$$

where $\bar{c}(t;\tau) \triangleq b(t)b(t+\tau)c_{2w_2}(\tau)$, has *vanishing* Fourier coefficients due to the finite support of $b(t)$. Hence, although $x(t)$ is not second-order cyclostationary (in the sense of (1), due to $\bar{c}(t;\tau)$), it exhibits *presence* of second-order cyclostationarity with cycles $\mathcal{A}_2 = \{-2\omega_0, 0, 2\omega_0\}$. □

An increased power test could be performed by using different τ's as well as different α's in $\hat{\mathbf{c}}^{(T)}_{2x}$. Although computationally more expensive, this is particularly useful in an exhaustive search for absence of cyclostationarity.

A simplified version of the given test, which is of particular interest in determining the cycles present in $\mathcal{C}_{2x}(\alpha;\tau)$ for a given τ, results by setting $N=1$. The test statistic and its implementation in practice is given as:

$$\bar{\mathcal{T}}_{2c} = T\hat{\mathbf{c}}_{2x}\hat{\mathbf{\Sigma}}^{-1}_{2c}\hat{\mathbf{c}}'_{2x} \tag{24}$$

where $\hat{\mathbf{c}}_{2x} = [\mathrm{Re}\{\frac{1}{T}\sum_{t=0}^{T-1}x(t)x(t+\tau)e^{-j\alpha t}\}, \mathrm{Im}\{\frac{1}{T}\sum_{t=0}^{T-1}x(t)x(t+\tau)e^{-j\alpha t}\}]$,

$$\hat{\mathbf{\Sigma}}_{2c}(m,n) = \begin{bmatrix} \mathrm{Re}\left\{\dfrac{\hat{\mathcal{S}}^{(T)}_{2f_{\tau,\tau}}(2\alpha;\alpha)+\hat{\mathcal{S}}^{(*T)}_{2f_{\tau,\tau}}(0;-\alpha)}{2}\right\} & \mathrm{Im}\left\{\dfrac{\hat{\mathcal{S}}^{(T)}_{2f_{\tau,\tau}}(2\alpha;\alpha)-\hat{\mathcal{S}}^{(*T)}_{2f_{\tau,\tau}}(0;-\alpha)}{2}\right\} \\ \mathrm{Im}\left\{\dfrac{\hat{\mathcal{S}}^{(T)}_{2f_{\tau,\tau}}(2\alpha;\alpha)+\hat{\mathcal{S}}^{(*T)}_{2f_{\tau,\tau}}(0;-\alpha)}{2}\right\} & \mathrm{Im}\left\{\dfrac{\hat{\mathcal{S}}^{(*T)}_{2f_{\tau,\tau}}(0;-\alpha)-\hat{\mathcal{S}}^{(T)}_{2f_{\tau,\tau}}(2\alpha;\alpha)}{2}\right\} \end{bmatrix} \tag{25}$$

with $\hat{\mathcal{S}}_{2f_{\tau,\tau}}(2\alpha;\alpha)$, $\hat{\mathcal{S}}^{(*)}_{2f_{\tau,\tau}}(0;-\alpha)$, and $F^{(T)}_{\tau}(\omega)$, as in (23). A *fast* implementation of the test results by observing that $F^{(T)}_{\tau}$ and $\hat{\mathbf{c}}_{2x}$ are essentially the periodograms of $x(t)x(t+\tau)$ and hence can be computed via FFT's. However, this limits the *resolution* (with reference to the search space of candidate α's) to the FFT grid. From (23), it follows that the choice of different $W^{(T)}$'s with different L's introduces different scale factors in (24).

It is possible to estimate the time-varying ensemble statistics of a cyclostationary process with the knowledge of the cycles, which are used, for example, in parametric identification methods [29]. Once the set of cycles $\mathcal{A}_2$ of $c_{2x}(t;\tau)$ has been estimated using the test in (24), one can estimate $c_{2x}(t;\tau)$ from (1) and (4) as

$$\hat{c}^{(T)}_{2x}(t;\tau) = \sum_{\alpha\in\mathcal{A}_2}\hat{\mathcal{C}}^{(T)}_{2x}(\alpha;\tau)e^{-j\alpha t}. \tag{26}$$

The estimator $\hat{c}^{(T)}_{2x}(t;\tau)$ can be shown to be mean-square sense consistent and asymptotically normal using the consistency and asymptotic normality of $\hat{\mathcal{C}}^{(T)}_{2x}(\alpha;\tau)$ [4].

In the next section, we generalize the time-domain tests for $k=2$ to estimate the set cycles, $\mathcal{A}_k$, of the kth-order cyclic-cumulant for $k\geq 1$. Along the same lines, consider the time-varying *mean* of $x(t)$, $c_{1x}(t) = E\{x(t)\} = \sum_{\alpha\in\mathcal{A}_1}\mathcal{C}_{1x}(\alpha)e^{j\alpha t}$, where $\mathcal{C}_{1x}(\alpha)$ is the *cyclic-mean.* Once the cycles of the mean, $\mathcal{A}_1$, are estimated, $c_{1x}(t)$ can be consistently estimated as $\hat{c}_{1x}(t) = \sum_{\alpha\in\mathcal{A}_1}\hat{\mathcal{C}}^{(T)}_{1x}(\alpha)\, e^{j\alpha t}$, where $\hat{\mathcal{C}}^{(T)}_{1x}(\alpha) = \frac{1}{T}\sum_{t=0}^{T-1}x(t)\,e^{-j\alpha t}$. Therefore, the zero-mean assumption about the cyclostationary process $x(t)$ does not sacrifice any generality, for if it is not zero-mean, one can estimate and subtract the mean.

An attempt was made in [33] to define a "frequency-decomposed measure of degree of cyclostationarity" for *continuous time* processes as

$$\mathrm{DCS}^\alpha \triangleq \frac{\int_{-\infty}^{\infty} \mid C_{2x}(\alpha;\tau) \mid^2 d\tau}{\int_{-\infty}^{\infty} \mid C_{2x}(0;\tau) \mid^2 d\tau} \tag{27}$$

where normalization in the denominator makes $0 \leq \mathrm{DCS}^\alpha \leq 1$. However neither a statistical test was provided to check for the "degree of cyclostationarity" nor the determination of the threshold of the probability of detection was given

III. Generalization to kth-Order

The test of the preceding section can be easily extended for checking the presence of cycles in the kth-order cyclic-cumulant of a cyclostationary process $x(t)$. This is useful for estimating cycles needed by the algorithms which employ higher than second-order statistics for processing cyclostationary signals [4], [5], [12], [27], [29]. Further, in certain situations it becomes necessary to deal with higher than second-order statistics, since second-order statistics are inadequate in providing information about the cycles. For example, in the transmission of QAM signals, the period (or equivalently, cycles) of the periodically time-varying channel *cannot* be inferred from the second-order statistics of its output alone. One has to employ a combination of second- and fourth-order cumulants to extract this information [29]. The following example considers a related situation and further motivates the need to consider the kth-order generalization.

Example 2: Consider a QAM signal $w(t) = a(t) + jb(t)$, where $a(t)$ and $b(t)$ are mutually independent, i.i.d., and assume binary values ± 1 with equal probability. Let $w(t)$ be transmitted on a mobile communication channel with (almost) periodically time-varying impulse response $h(t;\tau) = e^{j\omega_0 t} g(\tau)$. The received signal in the noise-free case is given as

$$\begin{aligned} y(t) &= \sum_{m=-\infty}^{\infty} h(t;m) w(t-m) \\ &= e^{j\omega_0 t} \sum_{m=-\infty}^{\infty} g(m) w(t-m). \end{aligned} \tag{28}$$

To compensate for intersymbol interference introduced due to the channel, one has to acquire knowledge of $h(t;\tau)$ by performing identification. Since the time variation of the channel is completely characterized by the exponential term $e^{j\omega_0 t}$, it is crucial to determine ω_0. Now, from the distribution of $a(t)$ and $b(t)$ it follows that $E\{a(t)\} \equiv E\{b(t)\} \equiv 0$, so that $E\{w(t)\} \equiv 0$ and one *cannot* estimate ω_0 from the cycle of $c_{1y}(t) \equiv 0$. Similarly

$$\begin{aligned} c_{2w}(t;\tau) &= E\{w(t)w(t+\tau)\} = \delta(\tau) E\{w^2(t)\} \\ &= \delta(\tau)\left[E\{a^2(t)\} - E\{b^2(t)\}\right] \equiv 0 \end{aligned} \tag{29}$$

where $\delta(\cdot)$ represents the Dirac delta. From (29) and (28) we see that $c_{2y}(t;\tau) \equiv 0$; therefore, the tests of the preceding section, although they correctly show that no cycles are present in $c_{2y}(t;\tau)$, *cannot* be used to estimate ω_0. Also, since $E\{w(t)w^*(t+\tau)\} = \delta(\tau)E\{|w(t)|^2\} = 2\delta(\tau)$, where $*$ denotes conjugation, it follows from (28) that $\bar{c}_{2y}(t;\tau) \triangleq E\{y(t)y^*(t+\tau)\} = 2e^{j\omega_0\tau}\sum_{m=-\infty}^{\infty} g(m)g^*(m+\tau)$. In general, one cannot determine ω_0 from $\bar{c}_{2y}(t;\tau)$ due to presence of the multiplicative factor $\sum_{m=-\infty}^{\infty} g(m)g^*(m+\tau)$.

From the preceding discussion it is evident that the second-order statistics (with or without conjugation) cannot be used to infer ω_0 and one must resort to higher than second-order statistics to obtain this information. Indeed for $k = 4$, since $m_{2y}(t;\tau) \equiv 0 \equiv m_{1y}(t)$, it follows from (11) and (28) that

$$\begin{aligned} c_{4y}(t;0,0,0) &\equiv m_{4y}(t;0,0,0) \\ &= -6e^{j4\omega_0 t} \sum_{m=-\infty}^{\infty} g^4(m) \end{aligned} \tag{30}$$

so that the cycle of $c_{4y}(t;0,0,0)$ (i.e., $4\omega_0$) yields an estimate of ω_0, motivating the need to develop tests for presence of cycles in kth-order cyclic-cumulants for $k \geq 2$.

The kth-order time-varying cumulant $c_{kx}(t;\tau)$ was defined in (11). A kth-order cyclostationary process is formally defined as a signal whose cumulants of order k are (almost) periodically time-varying [4], [6]. Thus, similar to (1), $c_{kx}(t;\tau)$ for each fixed $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_{k-1})$ can be expanded as a function of t in an FS as

$$\begin{aligned} c_{kx}(t;\boldsymbol{\tau}) &= \sum_{\alpha \in \mathcal{A}_k} \mathcal{C}_{kx}(\alpha;\boldsymbol{\tau}) e^{j\alpha t}, \\ \mathcal{A}_k &\triangleq \{\alpha\colon 0 \leq \alpha < 2\pi \text{ and } \mathcal{C}_{kx}(\alpha;\tau) \neq 0\} \end{aligned} \tag{31}$$

where the Fourier coefficients

$$\mathcal{C}_{kx}(\alpha;\boldsymbol{\tau}) \triangleq \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} c_{kx}(t;\boldsymbol{\tau}) e^{-j\alpha t} \tag{32}$$

are called the cyclic cumulants at cycle frequency α.

Our goal is to develop tests to find $\mathcal{A}_k$ of (31). If there exists one pair $(\alpha;\boldsymbol{\tau})$ for which $\mathcal{C}_{kx}(\alpha;\boldsymbol{\tau}) \neq 0$, then we say that kth-order cyclostationarity is present in $x(t)$. Hence, the kth-order tests of the subsequent sections also check for presence of kth-order cyclostationarity. As with the $k = 2$ case, this requires us to develop an estimator of $\mathcal{C}_{kx}(\alpha;\boldsymbol{\tau})$ along with its asymptotic properties.

A. Asymptotics of kth-Order Cyclic Statistics

The kth-order time-varying moment $m_{kx}(t;\boldsymbol{\tau}) \triangleq E\{x(t)\, x(t+\tau_1)\cdots x(t+\tau_{k-1})\}$ of a kth-order cyclostationary process is also (almost) periodically time-varying and accepts an FS [4], [6] as

$$\begin{aligned} m_{kx}(t;\boldsymbol{\tau}) &= \sum_{\alpha \in \mathcal{A}_k^m} \mathcal{M}_{kx}(\alpha;\boldsymbol{\tau}) e^{j\alpha t}; \\ \mathcal{M}_{kx}(\alpha;\boldsymbol{\tau}) &= \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} m_{kx}(t;\boldsymbol{\tau}) e^{-j\alpha t} \end{aligned} \tag{33}$$

where $\mathcal{M}_{kx}$ is the *cyclic-moment*, and $\mathcal{A}_k^m$ is the set of cycle frequencies of the moments.[2] Combining (11), (32), and (33),

[2]Note that the moment set of cycles $\mathcal{A}_k^m$ may be generally different from the cumulant set of cycles $\mathcal{A}_k$.

it follows that cyclic-cumulants can be expressed in terms of cyclic-moments [4]

$$\mathcal{C}_{kx}(\alpha;\boldsymbol{\tau}) = \sum_{\nu}(-1)^{p-1}(p-1)! \times \sum_{\alpha_1,\ldots,\alpha_p} \mathcal{M}_{\nu_1 x}(\alpha_1;\boldsymbol{\tau}_{\nu_1})\cdots\mathcal{M}_{\nu_p x}(\alpha_p;\boldsymbol{\tau}_{\nu_p}) \times \eta(\alpha-\alpha_1-\cdots-\alpha_p) \quad (34)$$

where $\eta(\alpha)$ is the Kronecker comb (train) function, which is nonzero and unity only when $\alpha = 0(\bmod 2\pi)$, and $\mathcal{M}_{\nu x}(\alpha;\boldsymbol{\tau}_\nu) \triangleq \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1} m_{\nu x}(t;\boldsymbol{\tau}_\nu)e^{-j\alpha t}$. Therefore, to develop an estimator of $\mathcal{C}_{kx}$ one needs an estimator for $\mathcal{M}_{kx}$. Motivated by (33), we define the estimator of $\mathcal{M}_{kx}$ (see also (4)) as

$$\hat{\mathcal{M}}_{kx}^{(T)}(\alpha;\boldsymbol{\tau}) \triangleq \frac{1}{T}\sum_{t=0}^{T-1} x(t)x(t+\tau_1)\cdots x(t+\tau_{k-1})e^{-j\alpha t}. \quad (35)$$

With $f(t;\boldsymbol{\tau}) \triangleq x(t)x(t+\tau_1)\cdots x(t+\tau_{k-1})$, defining the cyclic spectra of $f(t;\boldsymbol{\tau})$ as

$$\mathcal{S}_{2f_{\tau,\rho}}(\alpha;\omega) \triangleq \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\sum_{\xi=-\infty}^{\infty}\mathrm{cum}\{f(t;\boldsymbol{\tau}),f(t+\xi;\boldsymbol{\rho})\}e^{-j\omega\xi}e^{-j\alpha t}$$

$$\mathcal{S}_{2f_{\boldsymbol{\tau},\boldsymbol{\rho}}}^{(*)}(\alpha;\omega) \triangleq \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\sum_{\xi=-\infty}^{\infty}\mathrm{cum}\{f(t;\boldsymbol{\tau}),f^*(t+\xi;\boldsymbol{\rho})\}e^{-j\omega\xi}e^{-j\alpha t}$$

and generalizing proof Theorem 1, it can be shown that [4]

$$\lim_{T\to\infty}\hat{\mathcal{M}}_{kx}^{(T)}(\alpha;\boldsymbol{\tau}) \overset{m.s.s}{=} \mathcal{M}_{kx}(\alpha;\boldsymbol{\tau}). \quad (36)$$

In addition, $\sqrt{T}[\hat{\mathcal{M}}_{kx}^{(T)}(\alpha;\boldsymbol{\tau}) - \mathcal{M}_{kx}(\alpha;\boldsymbol{\tau})]$ is asymptotically complex normal with covariances given by

$$\lim_{T\to\infty} T\mathrm{cum}\left\{\hat{\mathcal{M}}_{kx}^{(T)}(\alpha;\boldsymbol{\tau}),\hat{\mathcal{M}}_{kx}^{(T)}(\beta;\boldsymbol{\rho})\right\} = \mathcal{S}_{2f_{\boldsymbol{\tau},\boldsymbol{\rho}}}(\alpha+\beta;\beta) \quad (37)$$

$$\lim_{T\to\infty} T\mathrm{cum}\{\hat{\mathcal{M}}_{kx}^{(T)}(\alpha;\tau),\hat{\mathcal{M}}_{kx}^{*(T)}(\beta;\rho)\} = \mathcal{S}_{2f_{\tau,\rho}}^{(*)}(\alpha-\beta;-\beta).$$

An estimator of $\mathcal{C}_{kx}$ can be obtained by substituting (35) into (34), for $\mathcal{M}_{kx}$. This estimator has been shown to be mean-square sense consistent and asymptotically normal with a computable variance using the asymptotic properties of $\hat{\mathcal{M}}_{kx}^{(T)}$ namely, (36) and (37). However, the general covariance expression for arbitrary k is complicated and can be found in [4].

In practice, it is usually sufficient to consider cumulants of order $k \leq 4$ and it follows from (34) and (35) that for zero-mean processes : $\hat{\mathcal{C}}_{3x}^{(T)}(\alpha;\tau_1,\tau_2) \equiv \hat{\mathcal{M}}_{3x}^{(T)}(\alpha;\tau_1,\tau_2)$, and (see Appendix B)

$$\hat{\mathcal{C}}_{4x}^{(T)}(\alpha;\tau_1,\tau_2,\tau_3) = \hat{\mathcal{M}}_{4x}^{(T)}(\alpha;\tau_1,\tau_2,\tau_3) - \sum_{\beta\in\mathcal{A}_2^m}\Big\{\hat{\mathcal{M}}_{2x}^{(T)}(\alpha-\beta;\tau_1)\hat{\mathcal{M}}_{2x}^{(T)}(\beta;\tau_3-\tau_2)e^{j\beta\tau_2} + \hat{\mathcal{M}}_{2x}^{(T)}(\alpha-\beta;\tau_2)\hat{\mathcal{M}}_{2x}^{(T)}(\beta;\tau_1-\tau_3)e^{j\beta\tau_3} + \hat{\mathcal{M}}_{2x}^{(T)}(\alpha-\beta;\tau_3)\hat{\mathcal{M}}_{2x}^{(T)}(\beta;\tau_2-\tau_1)e^{j\beta\tau_1}\Big\}. \quad (38)$$

Note that for $\forall\beta\in\mathcal{A}_2^m, \alpha-\beta\notin\mathcal{A}_2^m$, the terms in $\sum_{\beta\in\mathcal{A}_2^m}$ of (38), with cycles β and $(\alpha-\beta)$ vanish and hence

$$\mathcal{C}_{4x}(\alpha;\tau_1,\tau_2,\tau_3) \equiv \mathcal{M}_{4x}(\alpha;\tau_1,\tau_2,\tau_3) \quad (39)$$

which for this particular choice of cycles suggests a simpler estimator for $\mathcal{C}_{4x}$ when compared to (38). Notice that (39) holds regardless of α for QAM signals because $m_{2x}(t;\tau) \equiv 0 \equiv m_{1x}(t)$ (see also (30)). Apart from simplicity, another advantage of (39) is that the covariance expression of $\hat{\mathcal{M}}_{4x}$, obtained from (37) with $k = 4$, can be used to evaluate the covariance of $\hat{\mathcal{C}}_{4x}$. Thus, at least for zero-mean processes and for $k = 2, 3$, and 4, one can write

$$\lim_{T\to\infty} T\mathrm{cum}\left\{\hat{\mathcal{C}}_{kx}^{(T)}(\alpha;\boldsymbol{\tau}),\hat{\mathcal{C}}_{kx}^{(T)}(\beta;\boldsymbol{\rho})\right\} = \mathcal{S}_{2f_{\boldsymbol{\tau},\boldsymbol{\rho}}}(\alpha+\beta;\beta)$$

$$\lim_{T\to\infty} T\mathrm{cum}\left\{\hat{\mathcal{C}}_{kx}^{(T)}(\alpha;\boldsymbol{\tau}),\hat{\mathcal{C}}_{kx}^{*(T)}(\beta;\boldsymbol{\rho})\right\} = \mathcal{S}_{2f_{\boldsymbol{\tau},\boldsymbol{\rho}}}^{(*)}(\alpha-\beta;-\beta) \quad (40)$$

provided that (39) holds. With these comments we are prepared to present our kth-order time-domain test.

B. Test Statistic

Given a candidate frequency α and a set of lags $\tau_1,\ldots,\tau_N$, the problem of testing for presence of α in $[\mathcal{C}_{kx}(\alpha;\tau_1),\ldots,\mathcal{C}_{kx}(\alpha;\tau_N)]$ can be stated in a hypotheses-testing framework as done in (9):

$$\mathbf{H}_0{:}\alpha\notin\mathcal{A}_k\ \forall\{\tau_n\}_{n=1}^N \Longrightarrow \hat{\mathbf{c}}_{kx}^{(T)} = \boldsymbol{\epsilon}_{kx}^{(T)} \quad (41)$$

$$\mathbf{H}_1{:}\alpha\in\mathcal{A}_k \text{ for some } \{\tau_n\}_{n=1}^N \Longrightarrow \hat{\mathbf{c}}_{kx}^{(T)} = \mathbf{c}_{kx} + \boldsymbol{\epsilon}_{kx}^{(T)}$$

where $\hat{\mathbf{c}}_{kx}^{(T)} \triangleq [\mathrm{Re}\{\hat{\mathcal{C}}_{kx}^{(T)}(\alpha;\tau_1)\},\ldots,\mathrm{Re}\{\hat{\mathcal{C}}_{kx}^{(T)}(\alpha;\tau_N)\}, \mathrm{Im}\{\hat{\mathcal{C}}_{kx}^{(T)}(\alpha;\tau_1)\},\ldots,\mathrm{Im}\{\hat{\mathcal{C}}_{kx}^{(T)}(\alpha;\tau_N)\}]$, $\mathbf{c}_{kx} \triangleq [\mathrm{Re}\{\mathcal{C}_{kx}(\alpha;\tau_1)\},\ldots,\mathrm{Re}\{\mathcal{C}_{kx}(\alpha;\tau_N)\},\mathrm{Im}\{\mathcal{C}_{kx}(\alpha;\tau_1)\},\ldots,\mathrm{Im}\{\mathcal{C}_{kx}(\alpha;\tau_N)\},]$ and $\boldsymbol{\epsilon}_{kx}^{(T)}$ represents the estimation error. Using the asymptotic normality of cyclic-cumulant estimators we find that

$$\lim_{T\to\infty}\sqrt{T}\boldsymbol{\epsilon}_{kx}^{(T)} \overset{\mathcal{D}}{=} \mathcal{N}(\mathbf{0},\boldsymbol{\Sigma}_{kc}) \quad (42)$$

where $\boldsymbol{\Sigma}_{kc}$ is the asymptotic covariance matrix of the cyclic-cumulant estimators constructed analogous to the covariance of $k = 2$ case (17) by first filling in the entries of the $\mathbf{Q}$ matrices via (40) and then computing the $\boldsymbol{\Sigma}_{kc}$ matrix using (17) (see also the discussion (37)–(40)).

The test for presence of the kth-order cyclostationarity follows essentially the same steps as the $k = 2$ case, and therefore, we skip the details. The kth-order algorithm is given in Table I.

With $k = 3$ and $N = 1$, the algorithm of Table I yields the test for cycles in the third-order cumulant similar to the second-order case with $x(t)x(t+\tau)$ replaced by $x(t)x(t+\tau_1)x(t+\tau_2)$. It should be noted that for $k \geq 4$ the computation of sample cyclic-cumulants becomes complicated and burdensome with increasing order (34)–(38). On the other hand, the sample cyclic-polyspectrum estimators to be presented next have a simpler form.

TABLE I
TIME-DOMAIN TEST

Step 1	From data and using eqs. (37)-(38) compute [see also (42), (43) and (4)] $$\hat{\mathbf{c}}_{kx}^{(T)} = \left[Re\{\hat{C}_{kx}^{(T)}(\alpha;\boldsymbol{\tau}_1)\},\ldots,Re\{\hat{C}_{kx}^{(T)}(\alpha;\boldsymbol{\tau}_N)\},\ Im\{\hat{C}_{kx}^{(T)}(\alpha;\boldsymbol{\tau}_1)\},\ldots,Im\{\hat{C}_{kx}^{(T)}(\alpha;\boldsymbol{\tau}_N)\}\right].$$
Step 2	Fill in the entries of the covariance matrix $\hat{\boldsymbol{\Sigma}}_{kc}$ as discussed following eq. (46).
Step 3	Compute the value of the test statistic as $$\mathcal{T}_{kc} = T\ \hat{\mathbf{c}}_{kx}^{(T)}\ \hat{\boldsymbol{\Sigma}}_{kc}^{-1}\ \hat{\mathbf{c}}_{kc}^{(T)\prime}$$
Step 4	For a given probability of false alarm P_F, find, using the central χ^2 tables for $2N$ degrees of freedom (e.g., [20]), the threshold Γ such that $P_F = Pr\{\chi^2 \geq \Gamma\}$.
Step 5	Declare α as a cycle frequency at least for $\boldsymbol{\tau}_1,\ldots,\boldsymbol{\tau}_N$ if $\mathcal{T}_{kc} \geq \Gamma$; else decide that α is not a cycle frequency of $C_{kx}(\alpha;\boldsymbol{\tau})$ for any of $\boldsymbol{\tau}_1,\ldots,\boldsymbol{\tau}_N$.

IV. FREQUENCY-DOMAIN TEST

As mentioned in Section I, cyclostationarity can equally well be exploited using the frequency-domain counterpart of the cyclic-covariance called the cyclic-spectrum. Nonparametric algorithms for cyclostationary signals are typically based on the cyclic-spectra and assume knowledge of cycle frequencies present in the statistics of interest [1], [5], [10]. In this section we develop frequency-domain tests for cycle frequencies present in the cyclic-spectra and presence of cyclostationarity. Since the second-and kth-order tests follow the same steps, we directly present the kth-order case.

As with the spectra (3), the time-varying and cyclic-polyspectra are defined, respectively, as

$$S_{kx}(t;\boldsymbol{\omega}) \triangleq \sum_{\boldsymbol{\tau}=-\infty}^{\infty} c_{kx}(t;\tau)e^{-j\boldsymbol{\omega}\boldsymbol{\tau}'};$$
$$\mathcal{S}_{kx}(\alpha;\boldsymbol{\omega}) \triangleq \sum_{\boldsymbol{\tau}=-\infty}^{\infty} \mathcal{C}_{kx}(\alpha;\tau)e^{-j\boldsymbol{\omega}\boldsymbol{\tau}'} \tag{43}$$

where $\boldsymbol{\omega} \triangleq (\omega_1,\ldots,\omega_{k-1})$. Using (31) and (43), one can write

$$S_{kx}(t;\boldsymbol{\omega}) \triangleq \sum_{\alpha\in\mathcal{A}_k} \mathcal{S}_{kx}(\alpha;\boldsymbol{\omega})e^{j\alpha t} \tag{44}$$

which represents the FS expansion of the time-varying polyspectra. For a given $\boldsymbol{\omega}$, our aim here is to develop tests to find $\mathcal{A}_k$ i.e., all the α's for which $\mathcal{S}_{kx}(\alpha;\boldsymbol{\omega}) \neq 0$, and analogous to the time-domain tests, this requires estimators of $\mathcal{S}_{kx}(\alpha;\boldsymbol{\omega})$ along with their asymptotic distributions.

A. Asymptotics of Cyclic-Polyspectral Estimators

Notice from (3) and (43) that the cyclic-spectra and polyspectra are nothing but the Fourier transforms of the cyclic-cumulants. In this sense, they are similar to the spectra and polyspectra of stationary processes, which are Fourier transforms of the time-invariant cumulants. It therefore seems natural to modify the (poly) periodogram estimators [2] to estimate cyclic-(poly)spectra.

With $X_T(\omega) \triangleq \sum_{t=0}^{T-1} x(t)e^{-j\omega t}$, the *cyclic*-periodogram and bi-periodogram are defined, respectively, as

$$I_{2x}^{(T)}(\alpha;\omega) = \frac{1}{T}X_T(\omega)X_T(\alpha-\omega), \tag{45}$$
$$I_{3x}^{(T)}(\alpha;\omega_1,\omega_2) = \frac{1}{T}X_T(\omega_1)X_T(\omega_2)X_T(\alpha-\omega_1-\omega_2). \tag{46}$$

Notice that for $\alpha = 0$, $I_{2x}^{(T)}$ and $I_{3x}^{(T)}$ of (45) and (46) reduce to the conventional periodogram and bi-periodogram respectively, used for stationary processes. The kth-order cyclic-periodogram can be defined as

$$I_{kx}^{(T)}(\alpha;\omega_0,\ldots,\omega_{k-1}) \triangleq \frac{1}{T}X_T(\omega_0)\cdots X_T(\omega_{k-1}) \times \eta_{\alpha-\omega_0-\cdots-\omega_{k-1}} \tag{47}$$

where η denotes the Kronecker delta train, and hence, $I_{kx}^{(T)}$ is a $(k-1)$-dimensional function which is nonzero and unity only when $\omega_0+\cdots+\omega_{k-1} \equiv \alpha(\text{mod}\,2\pi)$.

Analogous to the stationary case the periodograms are unbiased but inconsistent estimators of the cyclic polyspectra, and spectral smoothing is needed to make them consistent [4], [6]. The smoothed periodogram estimate of the cyclic-polyspectra is given as

$$\begin{aligned}\hat{\mathcal{S}}_{kx}^{(T)}(\alpha;\omega_0,\ldots,\omega_{k-1}) &\triangleq T^{-k+1}\sum_{s_0,\ldots,s_{k-1}=0}^{T-1} \\ &\times W^{(T)}\left(\omega_0-\frac{2\pi s_0}{T},\ldots,\omega_{k-1}-\frac{2\pi s_{k-1}}{T}\right) \\ &\times I_{kx}^{(T)}\left(\alpha;\frac{2\pi s_0}{T},\ldots,\frac{2\pi s_{k-1}}{T}\right) \\ &\times \phi\left(\frac{2\pi s_0}{T},\ldots,\frac{2\pi s_{k-1}}{T}\right)\end{aligned} \tag{48}$$

where $W^{(T)}$ represents the spectral smoothing window and the function ϕ is defined as

$$\phi(\omega_0,\ldots,\omega_{k-1}) \triangleq \begin{cases}1, & \sum_{l\in\mathcal{L}}\omega_l \notin \mathcal{A}_{|\mathcal{L}|} \text{ and } \sum_{n=0}^{k-1}\omega_n = \alpha \\ 0, & \text{else}\end{cases}$$

where $\mathcal{L}$ are all the nonempty subsets of $\{1,\ldots,k-1\}$ with $|\mathcal{L}|$ elements, and its role is to suppress the contributions from the proper submanifolds [4], [6].

In practice the cyclic-spectra and bispectra estimators can be obtained by simplifying (48) for zero-mean processes as

$$\hat{\mathcal{S}}_{2x}^{(T)}(\alpha;\omega) \triangleq \frac{1}{TL}\sum_{s=-(L-1)/2}^{(L-1)/2} W^{(T)}(s) \times I_{2x}^{(T)}\left(\alpha;\omega-\frac{2\pi s}{T}\right), \quad (49)$$

$$\hat{\mathcal{S}}_{3x}^{(T)}(\alpha;\omega_1,\omega_2) \triangleq \frac{1}{TL^2}\sum_{s_1,s_2=-(L-1)/2}^{(L-1)/2} W^{(T)}(s_1)W^{(T)}(s_2) \times I_{3x}^{(T)}\left(\alpha;\omega_1-\frac{2\pi s_1}{T},\omega_2-\frac{2\pi s_2}{T}\right) \quad (50)$$

where $W^{(T)}$ is a spectral window of support L (odd).

Our interest is on the asymptotic properties of the cyclic-polyspectral estimators, which are presented next. If the window $W^{(T)}$ satisfies certain regularity conditions (usually met in practice by smooth windows), $x(t)$ satisfies **A1**, and $\sum_{\boldsymbol{\tau}} |\tau_i|\ |\hat{\mathcal{C}}_{kx}^{(T)}(\alpha;\boldsymbol{\tau})| < \infty$, $i=1,\ldots,k-1$, $\forall\alpha$, $\forall k$, then, $\hat{\mathcal{S}}_{kx}^{(T)}(\alpha;\omega_0,\ldots,\omega_{k-1})$ of (48) is mean-square sense consistent [4], [6], i.e.

$$\lim_{T\to\infty} \hat{\mathcal{S}}_{kx}^{(T)}(\alpha;\omega_0,\ldots,\omega_{k-1}) \overset{m.s.s.}{=} \mathcal{S}_{kx}(\alpha;\boldsymbol{\omega}). \quad (51)$$

Additionally, $\hat{\mathcal{S}}_{kx}^{(T)}$ is asymptotically complex normal with covariance given by

$$\begin{aligned}&\lim_{T\to\infty} B_T^{k-1}T \\ &\times \operatorname{cum}\left\{\hat{\mathcal{S}}_{kx}^{(T)}(\alpha;\omega_0,\ldots,\omega_{k-1}),\hat{\mathcal{S}}_{kx}^{(T)}(\beta;\mu_0,\ldots,\mu_{k-1})\right\} \\ &= \mathcal{E}_w \sum_p \eta\left(\alpha-\textstyle\sum_{l=0}^{k-1}\omega_l\right)\eta\left(\beta-\textstyle\sum_{l=0}^{k-1}\mu_l\right) \\ &\times \mathcal{S}_{2x}(\omega_0+\mu_{P(0)};+\mu_{P(0)}) \\ &\cdots \mathcal{S}_{2x}(\omega_{k-1}+\mu_{P(k-1)};+\mu_{P(k-1)})\end{aligned} \quad (52)$$

where $\mathcal{E}_w$ represents the energy of the window, P represents all the permutation of the integers $0,\ldots,k-1$ and B_T is the bandwidth of the spectral window [4], [6]. The conjugated covariance can be obtained by replacing μ with $-\mu$ and β with $-\beta$ in (52). Note that although the covariance expression of (52) seems complicated, it consists only of second-order cyclic-spectra which can be easily estimated using (49), irrespective of k. In practice, the covariance of the cyclic-spectra and bispectra can be computed using (52) and (49) as

$$\begin{aligned}&\widehat{\operatorname{cum}}\left\{\hat{\mathcal{S}}_{2x}^{(T)}(\alpha;\omega),\hat{\mathcal{S}}_{2x}^{(T)}(\beta;\mu)\right\} \\ &\approx \frac{\mathcal{E}_w}{TL}\Big\{\hat{\mathcal{S}}_{2x}^{(T)}(\omega+\mu;+\mu)\hat{\mathcal{S}}_{2x}^{(T)}(\alpha+\beta-\omega-\mu;\beta-\mu) \\ &+\hat{\mathcal{S}}_{2x}^{(T)}(\omega+\beta-\mu;\beta-\mu)\hat{\mathcal{S}}_{2x}^{(T)}(\alpha-\omega+\mu;\mu)\Big\}\end{aligned} \quad (53)$$

and with $\mu_2=\beta-\mu_0-\mu_1$ and $\omega_2=\alpha-\omega_0-\omega_1$

$$\begin{aligned}&\frac{L^2T}{\mathcal{E}_w}\widehat{\operatorname{cov}}\left\{\hat{\mathcal{S}}_{3x}^{(T)}(\alpha;\omega_0,\omega_1,\omega_2),\hat{\mathcal{S}}_{3x}^{(T)}(\beta;\mu_0,\mu_1,\mu_2)\right\} \\ &\approx \hat{\mathcal{S}}_{2x}^{(T)}(\omega_0+\mu_0;\mu_0)\hat{\mathcal{S}}_{2x}^{(T)}(\omega_1+\mu_1;\mu_1)\hat{\mathcal{S}}_{2x}^{(T)}(\omega_2+\mu_2;\mu_2) \\ &+\hat{\mathcal{S}}_{2x}^{(T)}(\omega_0+\mu_0;\mu_0)\hat{\mathcal{S}}_{2x}^{(T)}(\omega_1+\mu_2;\mu_2)\hat{\mathcal{S}}_{2x}^{(T)}(\omega_2+\mu_1;\mu_1) \\ &+\hat{\mathcal{S}}_{2x}^{(T)}(\omega_0+\mu_1;\mu_1)\hat{\mathcal{S}}_{2x}^{(T)}(\omega_1+\mu_0;\mu_0)\hat{\mathcal{S}}_{2x}^{(T)}(\omega_2+\mu_2;\mu_2) \\ &+\hat{\mathcal{S}}_{2x}^{(T)}(\omega_0+\mu_1;\mu_1)\hat{\mathcal{S}}_{2x}^{(T)}(\omega_1+\mu_2;\mu_2)\hat{\mathcal{S}}_{2x}^{(T)}(\omega_2+\mu_0;\mu_0) \\ &+\hat{\mathcal{S}}_{2x}^{(T)}(\omega_0+\mu_2;\mu_2)\hat{\mathcal{S}}_{2x}^{(T)}(\omega_1+\mu_1;\mu_1)\hat{\mathcal{S}}_{2x}^{(T)}(\omega_2+\mu_0;\mu_0) \\ &+\hat{\mathcal{S}}_{2x}^{(T)}(\omega_0+\mu_2;\mu_2)\hat{\mathcal{S}}_{2x}^{(T)}(\omega_1+\mu_0;\mu_0)\hat{\mathcal{S}}_{2x}^{(T)}(\omega_2+\mu_1;\mu_1).\end{aligned} \quad (54)$$

Recall that the covariance of the cyclic-cumulant estimator, $\mathcal{S}_{2f_{\tau},\tau}$ (14), (40) is the cyclic-spectrum $f(t;\tau)$ and therefore can be estimated using (48) with $k=2$. Strictly speaking, this requires inclusion of the 0–1 function $\phi(\omega)$, for avoiding the proper sub-manifolds which in the case of cyclic-spectra are the cycles of the *mean* of $f(t;\tau)$ i.e., $\mathcal{A}_k$, but since when testing for presence of cycles, $\mathcal{A}_k$ is unknown, we had set $\phi(\omega)\equiv 1$, except at the candidate cycle (24). The resulting error is negligible in practice and one may ignore it. Our simulation results agree with this conclusion.

Due to the consistency and asymptotic normality of the cyclic-polyspectral estimators the asymptotic properties of the frequency-domain and the time-domain statistics are completely analogous. Therefore, the steps involved in deriving the test of Section II can equally well be applied to develop the frequency-domain algorithm, as is done next.

B. Test Statistic

For a fixed $\boldsymbol{\omega}$ and a given candidate frequency α let

$$\hat{\boldsymbol{s}}_{kx}^{(T)} \triangleq \left[\operatorname{Re}\left\{\hat{\mathcal{S}}_{kx}^{(T)}(\alpha;\boldsymbol{\omega}_1)\right\},\ldots,\operatorname{Re}\left\{\hat{\mathcal{S}}_{kx}^{(T)}(\alpha;\boldsymbol{\omega}_N)\right\}, \operatorname{Im}\left\{\hat{\mathcal{S}}_{kx}^{(T)}(\alpha;\boldsymbol{\omega}_1)\right\},\ldots,\operatorname{Im}\left\{\hat{\mathcal{S}}_{kx}^{(T)}(\alpha;\boldsymbol{\omega}_N)\right\}\right] \quad (55)$$

whose asymptotic (true) value is given by

$$\boldsymbol{s}_{kx} \triangleq [\operatorname{Re}\{\mathcal{S}_{kx}(\alpha;\omega_1)\},\ldots,\operatorname{Re}\{\mathcal{S}_{kx}(\alpha;\omega_N)\}, \operatorname{Im}\{\mathcal{S}_{kx}(\alpha;\omega_1)\},\ldots,\operatorname{Im}\{\mathcal{S}_{kx}(\alpha;\omega_N)\}] \quad (56)$$

so that

$$\hat{\boldsymbol{s}}_{kx}^{(T)} = \boldsymbol{s}_{kx} + \boldsymbol{\epsilon}_{kx}^{(T)} \quad (57)$$

where $\boldsymbol{\epsilon}_{kx}^{(T)}$ is the estimation error. Using the asymptotic normality of cyclic-polyspectrum estimators, the error converges in distribution to a Gaussian density given by

$$\lim_{T\to\infty}\sqrt{TB_T^{k-1}}\boldsymbol{\epsilon}_{kx}^{(T)} \overset{\mathcal{D}}{=} \mathcal{N}(\mathbf{0},\boldsymbol{\Sigma}_{ks}) \quad (58)$$

where $\boldsymbol{\Sigma}_{ks}$ is the asymptotic covariance matrix, which is constructed in the same fashion as (17), by first computing the **Q** matrices with conjugated and unconjugated covariance of the polyspectral estimators and then using them to fill the $\boldsymbol{\Sigma}_{ks}$ matrix.

The hypotheses-testing problem corresponding to the detection of the cycle frequencies in $\mathcal{S}_{kx}$ may be stated as

TABLE II
FREQUENCY-DOMAIN TEST

Step 1	From data and using eq. (52) compute [see also (54-55)] $$\hat{\mathbf{s}}_{kx}^{(T)} = \left[Re\{\hat{\mathcal{S}}_{kx}^{(T)}(\alpha;\boldsymbol{\omega}_1)\},\ldots,Re\{\hat{\mathcal{S}}_{kx}^{(T)}(\alpha;\boldsymbol{\omega}_N)\},\ Im\{\hat{\mathcal{S}}_{kx}^{(T)}(\alpha;\boldsymbol{\omega}_1)\},\ldots,Im\{\hat{\mathcal{S}}_{kx}^{(T)}(\alpha;\boldsymbol{\omega}_N)\}\right].$$
Step 2	Fill in the entries of the covariance matrix $\boldsymbol{\Sigma}_{ks}$ using as discussed following eq. (63).
Step 3	Compute the value of the test statistic as $$\mathcal{T}_{ks} = T\ \hat{\mathbf{s}}_{kx}^{(T)}\ \hat{\boldsymbol{\Sigma}}_{ks}^{-1}\ \hat{\mathbf{s}}_{kx}^{(T)\prime}$$
Step 4	For a given probability of false alarm, P_F, find, using the central χ^2 tables for $2N$ degrees of freedom (e.g., [20]), the threshold Γ such that $P_F = Pr\{\chi^2 \geq \Gamma\}$.
Step 5	Declare α as a cycle frequency at least for $\boldsymbol{\omega}_1,\ldots,\boldsymbol{\omega}_N$ if $\mathcal{T}_{ks} \geq \Gamma$ else decide that α is not a cycle frequency of $\mathcal{S}_{kx}(\alpha;\boldsymbol{\omega})$ for any of $\boldsymbol{\omega}_1,\ldots,\boldsymbol{\omega}_N$.

follows:

$$\mathbf{H}_0\colon \alpha \notin \mathcal{A}_k\ \forall\ \{\boldsymbol{\tau}_n\}_{n=1}^N \Longrightarrow \hat{\boldsymbol{s}}_{kx}^{(T)} = \boldsymbol{\epsilon}_{kx}^{(T)} \tag{59}$$
$$\mathbf{H}_1\colon \alpha \in \mathcal{A}_k \text{ for some } \{\boldsymbol{\tau}_n\}_{n=1}^N \Longrightarrow \hat{\boldsymbol{s}}_{kx}^{(T)} = \boldsymbol{s}_{ks} + \boldsymbol{\epsilon}_{kx}^{(T)}$$

and the test statistic is given by

$$\mathcal{T}_{ks} = \hat{\boldsymbol{s}}_{kx}^{(T)}\hat{\boldsymbol{\Sigma}}_{ks}^{-1}\hat{\boldsymbol{s}}_{kx}^{(T)\prime}. \tag{60}$$

Following the steps of Section II, we arrive at the frequency-domain algorithm, which is summarized in Table II.

As in the time-domain algorithm, once the threshold Γ is set, one can approximately evaluate the probability of detection, $P_D \triangleq \Pr\{\mathcal{T}_{ks} \mid \mathbf{H}_1\}$. For large enough data lengths we may approximately write

$$\mathcal{T}_{ks} \sim \mathcal{N}(T\boldsymbol{s}_{kx}\boldsymbol{\Sigma}_{ks}^{-1}\boldsymbol{s}_{kx}', 4T\boldsymbol{s}_{kx}\boldsymbol{\Sigma}_{ks}^{-1}\boldsymbol{s}_{kx}') \tag{61}$$

Therefore P_D can be evaluated by substituting for $\boldsymbol{s}_{kx}$ and $\boldsymbol{\Sigma}_{ks}$ in (61) by their estimates and using the standard normal tables [20].

An interesting test was given in [19] which, in our notation and for FFT frequencies, uses the following function to derive the detection statistic

$$\gamma(\omega_1,\omega_2,M) \triangleq \frac{\left|\frac{1}{M}\sum_{m=0}^{M-1} X_T\left(\omega_1+\frac{2\pi m}{T}\right)X_T^*\left(\omega_2+\frac{2\pi n}{T}\right)\right|^2}{\frac{1}{M}\sum_{m=0}^{M-1}\left|X_T\left(\omega_1+\frac{2\pi m}{T}\right)\right|^2\frac{1}{M}\sum_{m=0}^{M-1}\left|X_T\left(\omega_2+\frac{2\pi m}{T}\right)\right|^2}. \tag{62}$$

The numerator is a sample correlation of the finite Fourier transform $X_T(\omega)$, used to measure the spectral correlation. Intuitively, this suggests that large values of M should improve the measure. The denominator serves as a normalizing factor. The authors in [19] provide the distribution of $\gamma(\omega_1,\omega_2,M)$ as

$$\Pr(\gamma > \Gamma) = (1-\Gamma)^{M-1} \tag{63}$$

which was derived by Goodman [15] under the assumption that $X_T(\omega)$ are zero-mean complex Gaussian with $E\{X_T(\omega_1+\frac{2\pi m}{T})\}$ and $E\{X_T(\omega_2+\frac{2\pi m}{T})\}$, being constants for $m=0,\ldots,M-1$, suggesting that one must have $M <<< T$ for (63) to be reasonably accurate. However, the "coherent" statistic [19] is derived by choosing $M = T$, which yields via Parseval's theorem

$$\gamma(0,\alpha,T) = \frac{\left|\frac{1}{T}\sum_{t=0}^{T-1}|x(t)|^2 e^{-j\frac{2\pi\alpha t}{T}}\right|^2}{\left[\frac{1}{T}\sum_{t=0}^{T-1}\left|x(t)\right|^2\right]^2}. \tag{64}$$

Apart from violating the assumption required to derive (63), the coherent statistic loses phase information in $x(t)$ which in fact may contain the cyclostationary component, as in $x(t) = w(t)\exp(-j\omega_0 t)$, where $w(t)$ is a stationary process. As an alternative to the coherent statistic, an "incoherent" statistic was derived in [19], which is given as

$$\delta(\alpha,M) = \frac{1}{L+1} \times \sum_{p=0}^{L}\gamma\left(\frac{2\pi pM}{T},\frac{2\pi(pM+\alpha)}{T},M\right). \tag{65}$$

However, the distribution of $\delta(\alpha,M)$, which is necessary for performing a statistical test, was not derived. A threshold setting based on (63) yields a Γ for a given probability of false alarm P_F as $\Gamma = 1 - P_F^{[1/(M-1)]}$, which approaches 0 exponentially as M increases. This can be detrimental to the performance of the test, especially if the variance of the test statistic decays slower than exponentially. Unlike the kth-order treatment herein, the test of [19] considers only correlations of periodically correlated process.

V. SIMULATIONS

A modulating signal $w(t)$ was generated by passing zero-mean exponential deviates through an all-pole filter with poles at $0.45 \pm j0.35$. The data $x(t)$ was generated as

$$x(t) = w(t)\cos(\omega_0 t); \omega_0 = \pi/4. \tag{66}$$

The signal $x(t)$ models AM signals, or can be used to model sinusoids with randomly fluctuating amplitudes [13]. Our goal is to estimate the cycles present in the kth-order cyclic statistics of $x(t)$ for $k = 2$ and 3, using the time- and

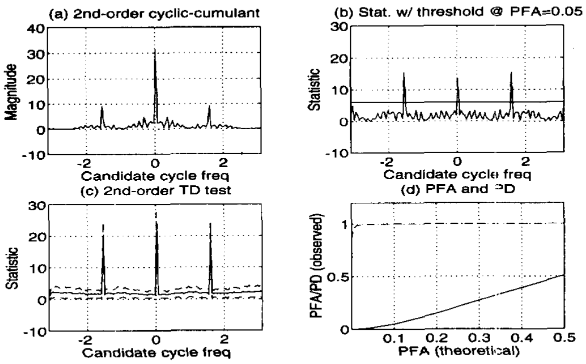

Fig. 1. Second-order time-domain tests.

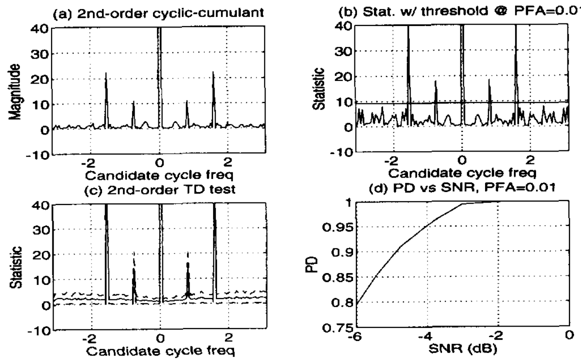

Fig. 2. Second-order time-domain tests—two signals.

frequency-domain algorithms. Using (1), (3), (31), (32), (43), and (44), it follows that

$$c_{2x}(t;\tau) = \frac{c_{2w}(\tau)}{4} e^{j\omega_0\tau} e^{-j2\omega_0 t} + \frac{c_{2w}(\tau)}{2}\cos\omega_0\tau + \frac{c_{2w}(\tau)}{4} e^{j\omega_0\tau} e^{j2\omega_0 t} \tag{67}$$

$$\begin{aligned} c_{3x}(t;\tau_1,\tau_2) = {} & \frac{c_{3w}(\tau_1,\tau_2)}{8} e^{j\omega_0(\tau_1+\tau_2)} e^{-j3\omega_0 t} + \frac{c_{3w}(\tau_1,\tau_2)}{8} \\ & \times \left[2\cos(\omega_0[\tau_1-\tau_2]) + e^{j\omega_0(\tau_1+\tau_2)}\right] e^{-j\omega_0 t} \\ & + \frac{c_{3w}(\tau_1,\tau_2)}{8} \\ & \times \left[2\cos(\omega_0[\tau_1-\tau_2]) + e^{j\omega_0(\tau_1+\tau_2)}\right] e^{j\omega_0 t} \\ & + \frac{c_{3w}(\tau_1,\tau_2)}{8} e^{j\omega_0(\tau_1+\tau_2)} e^{j3\omega_0 t} \end{aligned} \tag{68}$$

$$\begin{aligned} S_{2x}(t;\tau) = {} & \frac{S_{2w}(\omega-\omega_0)S_{2w}(\omega+\omega_0)}{4} \\ & + \frac{S_{2w}(\omega+\omega_0)}{4} e^{-j2\omega_0 t} + \frac{S_{2w}(\omega-\omega_0)}{4} e^{j2\omega_0 t} \end{aligned} \tag{69}$$

$$\begin{aligned} S_{3x}(t;\tau_1,\tau_2) = {} & \frac{S_{3w}(\omega_1-\omega_0,\omega_2-\omega_0)}{8} e^{-j3\omega_0 t} \\ & + \frac{S_{3w}(\omega_1-\omega_0,\omega_2-\omega_0)}{8} + \frac{S_{3w}(\omega_1-\omega_0,\omega_2+\omega_0)}{8} \\ & + \frac{S_{3w}(\omega_1+\omega_0,\omega_2+\omega_0)}{8} e^{-j\omega_0 t} \\ & + \frac{S_{3w}(\omega_1-\omega_0,\omega_2-\omega_0)}{8} \\ & + \frac{S_{3w}(\omega_1-\omega_0,\omega_2+\omega_0)}{8} + \frac{S_{3w}(\omega_1+\omega_0,\omega_2+\omega_0)}{8} e^{j\omega_0 t} \\ & + \frac{S_{3w}(\omega_1+\omega_0,\omega_2+\omega_0)}{8} e^{j3\omega_0 t}. \end{aligned} \tag{70}$$

From (67)–(70) it is evident that $\mathcal{A}_2 = \{-2\omega_0, 0, 2\omega_0\}$ and $\mathcal{A}_3 = \{-3\omega_0, -\omega_0, \omega_0, 3\omega_0\}$ at most, for all $\tau, \omega, \tau_1, \tau_2, \omega_1$ and ω_2, which is verified in the following simulations.

A. Second-Order Time-Domain Tests

The algorithm of Table I, with $k = 2$, was tested with $T = 256$ (128×2) and $\tau = 0$. A Kaiser window of parameter 10 was used to compute the covariance estimates in (24) with $L = 61$. Fig. 1(a) shows the magnitude of the "raw" second-order sample cyclic-cumulant of (4), from which it is seen that $\hat{C}_{2x}^{(T)}(\alpha;\tau)$ shows several peaks indicating the possibility of the presence of several cycles. Fig. 1(b) shows the test statistic (24) along with a threshold Γ set to CFAR of 0.05, which clearly shows that only three frequencies are statistically significant and they are $\{-\pi/2 = -1.57, 0, \pi/2 = 1.57\}$, as expected (66), (67). Fig. 1(c) shows mean $\pm$ standard deviation of the test statistic for 100 Monte Carlo runs. To verify the performance of the test we have plotted in Fig. 1(d) the theoretical probability of false alarms i) versus the probability of false alarms (PFA, solid line) and ii) the probability of detection (PD, dashed line), observed over 100 Monte Carlo runs. The probability of detection can be seen to rapidly approach the value 1, whereas the probability of false alarm is almost a straight line from 0 to 0.5, as expected.

B. Second-Order Time-Domain-2 Signals

To check the sensitivity of our test to the relative strengths of a superposition of two cyclostationary signals we picked $\omega_1 = \frac{\pi}{8}; \omega_2 = \frac{\pi}{4}$, and

$$x(t) = w_1(t)\cos(\omega_1 t) + w_2(t)\cos(\omega_2 t) \tag{71}$$

where $w_1(t)$ and $w_2(t)$ were two independent time-series generated by passing exponential deviates through the same filter as the $w(t)$ in (66). One of the two signals could be considered as an interference or noise. The variance of $w_2(t)$ was four times that of $w_1(t)$. The signal-to-noise-ratio SNR was defined as $10\log_{10}\left(\text{var}\{w_1(t)\}/\text{var}\{w_2(t)\}\right)$. The signal $x(t)$ was processed using the second-order time-domain test as in the previous experiment to detect the five cycle frequencies $\{-2\omega_2, -2\omega_1, 0, 2\omega_1, 2\omega_2\}$. The results are shown in Figs. 2(a), (b), and (c), which are analogous to those of Figs. 1(a), (b), and (c). It was seen that for better performance, we had to increase the data size to $T = 2048$ (128×16); otherwise, the estimation error was dominating the weaker signal. Further, for a fixed $T = 2048$ (128×16) and $P_F = 0.01$, we studied

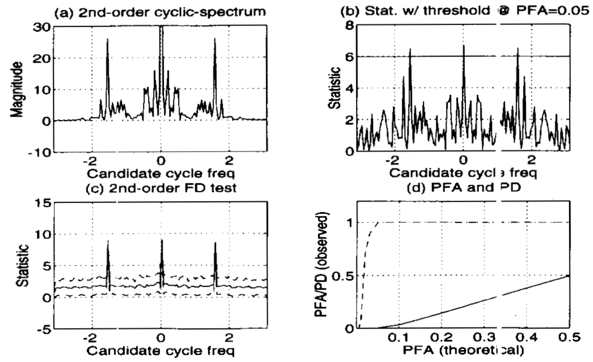

Fig. 3. Second-order frequency-domain tests.

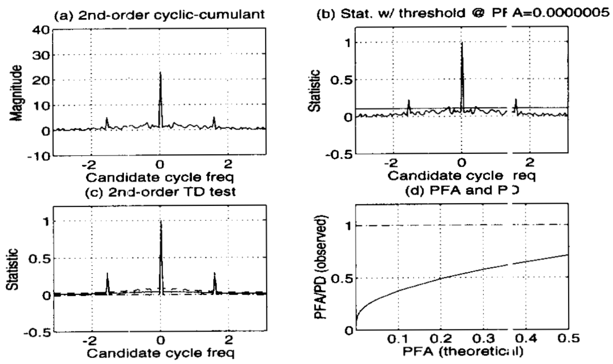

Fig. 4. Coherent statistic of [19].

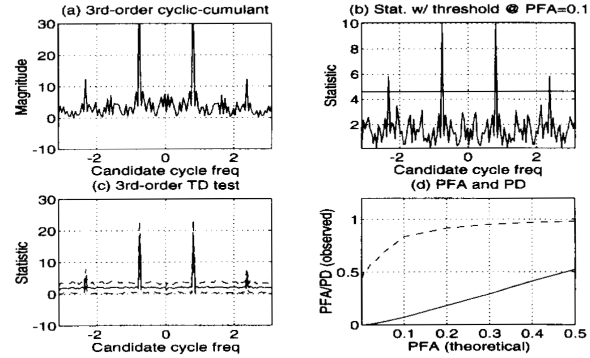

Fig. 5. Third-order time-domain tests.

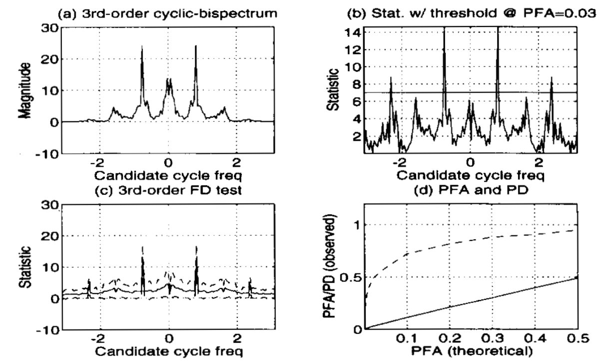

Fig. 6. Third-order frequency-domain tests.

the probability of detection for different values of the SNR; results are shown in Fig. 2(d).

C. Second-Order Frequency-Domain Tests

The frequency-domain algorithm of Table II, with $k = 2$, was tested with $T = 256$ (128×2) $\omega = 0$ and $x(t)$, as in (66). A Kaiser window with parameter 1 was used for computing the cyclic-spectrum as in (49) as well as the estimates of its covariance, as in (53) with $L = 11$. Fig. 3 shows the various plots analogous to the diagrams in Fig 1.

For comparison we have plotted the coherent statistics of [19] in Fig. 4 with $T = 256$ (128×2). Because the data length per segment is large $(T = 128)$, the threshold computed using (63) rapidly approaches zero as P_F increases (see discussion following (65)). As a consequence, the observed P_F consistently exceeds its theoretical value while the probability of detection is 1 since most of the frequencies exceed the small threshold value.

D. Third-Order Time-Domain Tests

The algorithm of Table I, with $k = 3$, was tested with $T = 2048$ (128×16) and $\tau_1 = \tau_2 = 0$. A Kaiser window with parameter 10 was employed for computing the covariance estimates with $L = 41$. Fig. 5 shows the various plots analogous to the ones in Fig. 1. It can be seen that the higher-order test requires more data than its second-order counterpart to reduce the increased variance caused by the higher order statistics.

E. Third-Order Frequency-Domain Tests

The algorithm of Table II, with $k = 3$ was tested with $T = 2048$ (128×16) and $\omega_1 = \omega_2 = 0$. A Kaiser window with parameter 15, $L = 31$ was employed for computing the cyclic-bispectrum while parameter 1, $L = 11$ was used for the estimates of its covariance. Note that because of increased variance of higher order statistics, longer window averaging lengths are required to compute the cyclic-bispectrum when compared to the ones required for variance estimation, which depend only upon the second-order spectra (52). Fig. 6 shows the various plots analogous to the ones in Fig. 1.

F. Target Motion Detection

As an application of the second-order algorithm we simulated a target motion detection in the radar scenario [1]. A cosinusoidal pulse is transmitted at frequency ω_0 and when the target is stationary we receive

$$x_s(t) = w(t)\cos(\omega_0 t) + v(t) \tag{72}$$

where $w(t)$ is a stationary process which models random effects introduced due to the target and the medium. When it is in motion with constant velocity, the received signal is

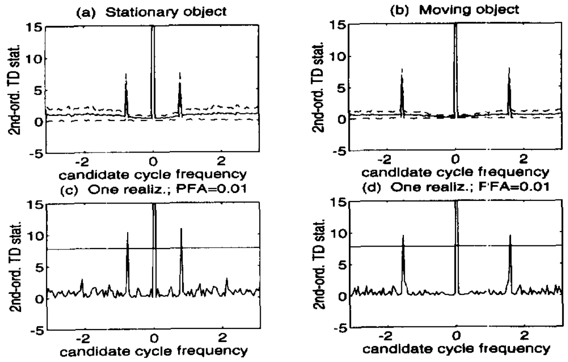

Fig. 7. Motion detection.

given as

$$x_m(t) = w(t)\cos(\omega_0 D[t-d]) + v(t) \tag{73}$$

where D accounts for the Doppler effect and d depicts the delay. It follows, using (1) and (3), that $\mathcal{A}_2 = \{-2\omega_0, 0, 2\omega_0\}$ when the object is stationary and $\mathcal{A}_2 = \{-2D\omega_0, 0, 2D\omega_0\}$ when the object is in motion. With $D = 2$, $d = 0$, $\omega_0 = \pi/8$, $v(t)$ to be a colored Gaussian noise (MA(16) approximation of an AR(2) filter with poles at $0.1 \pm j0.25$) and with $E\{w^2(t)\} = E\{v^2(t)\}$, we tested our second-order time-domain algorithm for $T = 1024$ (128×8), Kaiser window parameter 10, $L = 61$. The probability of false alarms was fixed at 0.01. Fig. 7(c) shows one realization of the statistic when the object is stationary, whereas Fig. 7(d) shows one realization of the statistic when the target is in motion. It can be seen that when the object moves the cycles appear at $\{-4\omega_0, 0, 4\omega_0\}$ instead of $\{-2\omega_0, 0, 2\omega_0\}$, correctly detecting motion. Fig. 7(a) and 7(b) shows mean and mean $\pm$ standard deviation of the second-order test statistics, over 100 Monte Carlo runs corresponding to Fig. 7(c) and (d), respectively.

VI. Conclusion

Presence or absence of kth-order cyclostationarity in a time series is defined by the presence or absence, respectively, of kth-order *cyclic*-cumulants and polyspectra in their corresponding time-varying ensemble averages. The main idea in the development of the tests was to establish that the kth-order sample cyclic-cumulants and polyspectra are consistent and asymptotically normal with computable variances, and therefore, asymptotic χ^2 tests could be developed for checking for (non)zeroness (presence or absence) of sample cyclic-cumulants or polyspectra. The variance normalization leads to a standardization of the thresholding process irrespective of k, time-, or frequency-domains. Simulations confirm the performance of the tests.

Implementation aspects, special cases, and explicit algorithms for $k \leq 4$ were discussed. Computationally, it seems that for $k \leq 3$, the time-domain tests are convenient to use; however, for $k \geq 4$, the frequency-domain tests are simpler to implement. Our tests are expected to be a necessary *first step* for gaining knowledge of the cycles, in the implementation of the algorithms which exploit cyclostationarity. The frequency-domain tests are appropriate for nonparametric algorithms which usually employ frequency-domain statistics, while the time-domain tests are suited for cyclic-cumulant based methods. Although fast FFT-based implementations of the tests are possible, they are limited by the resolution provided by the FFT's. In future it will be of interest to develop high resolution implementations to estimate cycles of a cyclostationary time series.

Appendix A
Proof of Theorem 1

Unbiasedness and Consistency: Asymptotic unbiasedness follows easily, since from (4) and (1)

$$\begin{aligned}\lim_{T\to\infty} E\left\{\hat{C}_{2x}^{(T)}(\alpha;\tau)\right\} &= \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1} E\{x(t)x(t+\tau)e^{-j\alpha t}\}\\ &= \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1} c_{2x}(t;\tau)e^{-j\alpha t}\\ &= C_{2x}(\alpha;\tau).\end{aligned} \tag{74}$$

For consistency, observe from (4) and (12) and the multilinearity of cumulants (see p. 19 of [2]) that

$$\begin{aligned}&\mathrm{cum}\left\{\hat{C}_{2x}^{(T)}(\alpha;\tau), \hat{C}_{2x}^{(T)}(\beta;\rho)\right\}\\ &= \frac{1}{T^2}\sum_{t_1,t_2=0}^{T-1}\mathrm{cum}\ \{f(t_1;\tau), f(t_2;\rho)\}e^{-j(\alpha t_1+\beta t_2)}\\ &= \frac{1}{T^2}\sum_{t=0}^{T-1}\sum_{\xi=-t}^{T-t-1}\mathrm{cum}\{f(t;\tau), f(t+\xi;\rho)\}e^{-j(\alpha+\beta)t}e^{-j\beta\xi}\end{aligned} \tag{75}$$

where $\xi \triangleq t_2 - t_1$ and $t \triangleq t_1$. Using the Leonov–Shiryaev identity (see p. 19 of [2]) to write the cumulant of products of $x(t)$ in (75) as a sum of products of its cumulants, we obtain

$$\mathrm{cum}\left\{\hat{C}_{2x}^{(T)}(\alpha;\tau), \hat{C}_{2x}^{(T)}(\beta;\rho)\right\} = \mathbf{T_1} + \mathbf{T_2} + \mathbf{T_3} \tag{76}$$

where

$$\begin{aligned}\mathbf{T}_1 &\triangleq \frac{1}{T^2}\sum_{t=0}^{T-1}\sum_{\xi=-t}^{T-t-1} c_{4x}(t;\tau,\xi,\xi+\rho)\\ &\quad\times e^{-j(\alpha+\beta)t}e^{-j\beta\xi};\\ \mathbf{T}_2 &\triangleq \frac{1}{T^2}\sum_{t=0}^{T-1}\sum_{\xi=-t}^{T-t-1} c_{2x}(t;\xi)\\ &\quad\times c_{2x}(t+\tau;\xi+\rho-\tau)e^{-j(\alpha+\beta)t}e^{-j\beta\xi};\\ \mathbf{T}_3 &\triangleq \frac{1}{T^2}\sum_{t=0}^{T-1}\sum_{\xi=-t}^{T-t-1} c_{2x}(t;\xi+\rho)\\ &\quad\times c_{2x}(t+\tau;\xi-\tau), e^{-j(\alpha+\beta)t}e^{-j\beta\xi}.\end{aligned} \tag{77}$$

We show that each of $\mathbf{T}_1, \mathbf{T}_2$ and $\mathbf{T}_3$ in (76) vanishes asymptotically due to **A1**. Now

$$\mathbf{T}_1 \leq \frac{1}{T^2}\sum_{t=0}^{T-1}\sum_{\xi=-\infty}^{\infty}|c_{4x}(t;\tau,\xi,\xi+\rho)|. \tag{78}$$

Using the summability of cumulants from **A1** with $k = 4$ we observe that $\sum_{\xi=-\infty}^{\infty} |c_{4x}(t;\tau,\xi,\xi+\rho)| = O(1)$, so that $\mathbf{T}_1$ in (78) is $O(T^{-1})$, hence, $\lim_{T\to\infty} \mathbf{T_1} = 0$. Next consider the second term from (76) and as with $\mathbf{T}_1$

$$\mathbf{T}_2 \leq \frac{1}{T^2} \sum_{t=0}^{T-1} \sum_{\xi=-\infty}^{\infty} |c_{2x}(t;\xi)| \times \sum_{\xi=-\infty}^{\infty} |c_{2x}(t+\tau;\xi+\rho-\tau)|.$$

Again, using **A**1 we observe that $\sum_{\xi=-\infty}^{\infty}|c_{2x}(t;\xi)| = O(1)$ and $\sum_{\xi=-\infty} |c_{2x}(t+\tau;\xi+\rho-\tau)| = O(1)$ so that $\mathbf{T}_2$ is $O(T^{-1})$, and hence, $\lim_{T\to\infty} \mathbf{T_2} = 0$. As with $\mathbf{T}_2$ it can be shown that $\mathbf{T}_3 = O(T^{-1})$ and hence $\lim_{T\to\infty} \mathbf{T}_3 = 0$. Since all the three terms in (76) vanish as $T \to \infty$, it follows that

$$\lim_{T\to\infty} \text{cum}\left\{\hat{C}_{2x}^{(T)}(\alpha;\tau), \hat{C}_{2x}^{(T)}(\beta;\rho)\right\} = 0, \ \forall \alpha,\beta,\tau,\rho. \quad (79)$$

Similarly, it can be shown that

$$\lim_{T\to\infty} \text{cum}\left\{\hat{C}_{2x}^{(T)}(\alpha;\tau), \hat{C}_{2x}^{*(T)}(\beta;\rho)\right\} = 0, \ \forall \alpha,\beta,\tau,\rho \quad (80)$$

proving the consistency[3] of $\hat{C}_{2x}^{(T)}(\alpha;\tau)$.

Asymptotic Normality We show asymptotic normality as in [2] (see also pp. 179–182 of [26]) by showing that cumulants of order ≥ 3 of $\hat{C}_{2x}^{(T)}(\alpha;\tau)$ (conjugated or unconjugated) vanish asymptotically. Using again (4) and (12) and the multilinearity of cumulants (see p. 19 of [2]), we find that

$$\text{cum}\left\{\hat{C}_{2x}^{(T)}(\alpha_0;\tau_0), \ldots, \hat{C}_{2x}^{(T)}(\alpha_m;\tau_m)\right\} = \frac{1}{T^{m+1}} \sum_{t_0,\ldots,t_m=0}^{T-1} \text{cum}\{f(t_0;\tau_0),\ldots,f(t_m;\tau_m)\} \times e^{-j(\alpha_0 t_0 + \cdots + \alpha_m t_m)}. \quad (81)$$

With $t_1 - t_0 = \xi_1, \ldots, t_m - t_0 = \xi_m$, and $t_0 = t$ in (81), it follows that

$$\text{cum}\left\{\hat{C}_{2x}^{(T)}(\alpha_0;\tau_0), \ldots, \hat{C}_{2x}^{(T)}(\alpha_m;\tau_m)\right\} = \frac{1}{T^{m+1}} \times \sum_{\xi=-(T-1)}^{(T-1)} \sum_{t=t_\alpha}^{t_\beta} \text{cum}\{f(t;\tau_0),\ldots,f(t+\xi_n;\tau_m)\} \times e^{-j\sum_{l=0}^{m}\alpha_l t} e^{-j\sum_{l=1}^{m}\alpha_l\xi_l} \quad (82)$$

where $t_\alpha \triangleq -\min(0,\xi_1,\ldots,\xi_m)$, $t_\beta = T - 1 - \max\{\xi_1,\ldots,\xi_m,0\}$. After using the Leonov–Shiryaev identity, once again we obtain that

$$\text{cum}\left\{\hat{C}_{2x}^{(T)}(\alpha_0;\tau_0), \ldots, \hat{C}_{2x}^{(T)}(\alpha_m;\tau_m)\right\} = \frac{1}{T^{m+1}} \times \sum_{\xi=-(T-1)}^{(T-1)} \sum_{t=t_\alpha}^{t_\beta} \sum_{\nu} \text{cum}\{x(\rho), \rho \in \nu_1\} \cdots \text{cum}\{x(\rho), \rho \in \nu_p\} e^{-j\sum_{l=0}^{m}\alpha_l t} e^{-j\sum_{l=1}^{m}\alpha_l\xi_l} \quad (83)$$

[3] Note that for consistency, **A**1 needs to hold only for $k = 2$ and 4.

where the summation on ν is over all the indecomposable partitions [2] of the following table:

$$\begin{matrix} t & t+\tau_0 \\ t+\xi_1 & t+\xi_1+\tau_1 \\ \vdots & \vdots \\ t+\xi_m & t+\xi_m+\tau_1 \end{matrix}. \quad (84)$$

As done with $\mathbf{T}_2$ of (76), it can be shown that the r.h.s. of (83) is $O(T^{-m})$ due to **A1**; Thus, from (83)

$$\lim_{T\to\infty} T^{m-1}\text{cum}\left\{\hat{C}_{2x}^{(T)}(\alpha_0;\tau_0), \ldots, \hat{C}_{2x}^{(T)}(\alpha_m;\tau_m)\right\} = 0. \quad (85)$$

Similarly, it can be shown that (85) holds even when any of the $\hat{C}_{2x}^{(T)}$'s are conjugated, which proves the asymptotic normality of $\hat{C}_{2x}^{(T)}(\alpha;\tau)$. The real and imaginary parts of $\hat{C}_{2x}^{(T)}(\alpha;\tau)$ are thus jointly Gaussian.

Covariance Expression: With $m = 1$ in (82) we obtain

$$\begin{aligned} &\text{cum}\left\{\hat{C}_{2x}^{(T)}(\alpha;\tau), \hat{C}_{2x}^{(T)}(\beta;\rho)\right\} \\ &= \frac{1}{T^2} \sum_{\xi=-(T-1)}^{(T-1)} \sum_{t=t_\alpha}^{t_\beta} \text{cum}\{f(t;\tau), f(t+\xi;\rho)\} e^{-j(\alpha+\beta)t} \times e^{-j\beta\xi} \\ &= \frac{1}{T^2} \sum_{\xi=-(T-1)}^{(T-1)} \left[\sum_{t=0}^{T-1} - \sum_{t=0}^{t_\alpha} - \sum_{t=t_\beta+1}^{T-1}\right] \times \text{cum}\{f(t;\tau), f(t+\xi;\rho)\} e^{-j(\alpha+\beta)t} e^{-j\beta\xi}. \end{aligned} \quad (86)$$

Observing that since $t_\alpha \triangleq -\min(0,\xi)$, $t_\beta = T - 1 - \max\{\xi,0\}$, for each fixed ξ, $\sum_{t=0}^{t_\alpha}$, and $\sum_{t=t_\beta+1}^{T-1}$ contain only a finite number of terms which vanish asymptotically when divided by T. It follows from (86) and (4) that:

$$\lim_{T\to\infty} T\text{cum}\left\{\hat{C}_{2x}^{(T)}(\alpha;\tau), \hat{C}_{2x}^{(T)}(\beta;\rho)\right\} = \mathcal{S}_{2f_{\tau,\rho}}(\alpha+\beta;\beta). \quad (87)$$

The assumption **A**1 guarantees that the sum on ξ converges at a rate that allows (87) to hold true [4]. It can be similarly shown that

$$\lim_{T\to\infty} T\ \text{cum}\left\{\hat{C}_{2x}^{(T)}(\alpha;\tau), \hat{C}_{2x}^{*(T)}(\beta;\rho)\right\} = \mathcal{S}_{2f_{\tau,\rho}}^{(*)}(\alpha-\beta;-\beta) \quad (88)$$

which completes the proof of the theorem.

APPENDIX B
THE FOURTH-ORDER CYCLIC-CUMULANT

From (11) with $k = 4$, it follows that for zero-mean processes

$$\begin{aligned} c_{4x}(t;\tau_1,\tau_2,\tau_3) = m_{4x}(t;\tau_1,\tau_2,\tau_3) &- m_{2x}(t;\tau_1)m_{2x}(t+\tau_2;\tau_3-\tau_2) \\ &- m_{2x}(t;\tau_2)m_{2x}(t+\tau_3;\tau_1-\tau_3) \\ &- m_{2x}(t;\tau_3)m_{2x}(t+\tau_1;\tau_2-\tau_1). \end{aligned} \quad (89)$$

Using (33) and the definition of cyclic-cumulants from (32) with $k = 4$, we obtain

$$\mathcal{C}_{4x}(\alpha;\tau_1,\tau_2,\tau_3) = \mathcal{M}_{4x}(\alpha;\tau_1,\tau_2,\tau_3) - \mathbf{M_1} - \mathbf{M_2} - \mathbf{M_3} \tag{90}$$

where

$$\mathbf{M_1} = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} m_{2x}(t;\tau_1)m_{2x}(t+\tau_2;\tau_3-\tau_2)e^{-j\alpha t} \tag{91}$$

$$\mathbf{M_2} = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} m_{2x}(t;\tau_2)m_{2x}(t+\tau_3;\tau_1-\tau_3)e^{-j\alpha t} \tag{92}$$

$$\mathbf{M_3} = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} m_{2x}(t;\tau_3)m_{2x}(t+\tau_1;\tau_2-\tau_1)e^{-j\alpha t}. \tag{93}$$

Now consider $\mathbf{M_1}$ in (91) and express $m_{2x}(t;\tau)$ via (33) to see that

$$\mathbf{M_1} = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1}\sum_{\beta\in\mathcal{A}_2^m}\sum_{\psi\in\mathcal{A}_2^m} \mathcal{M}_{2x}(\psi;\tau_1) \times \mathcal{M}_{2x}(\beta;\tau_3-\tau_2)e^{j\psi t}e^{j\beta(t+\tau_2)}e^{-j\alpha t}. \tag{94}$$

Passing the limit along with the summation on t in (94) inside the summations on β and ψ and observing that $\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1} e^{j\theta t} = \eta(\theta)$, a Kronecker delta train with period 2π, we obtain

$$\mathbf{M_1} = \sum_{\beta\in\mathcal{A}_2^m}\sum_{\psi\in\mathcal{A}_2^m} \mathcal{M}_{2x}(\psi;\tau_1) \times \mathcal{M}_{2x}(\beta;\tau_3-\tau_2)\eta(\alpha-\psi-\beta)e^{j\beta\tau_2} \tag{95}$$

$$= \sum_{\beta\in\mathcal{A}_2^m} \mathcal{M}_{2x}(\alpha-\beta;\tau_1)\mathcal{M}_{2x}(\beta;\tau_3-\tau_2)e^{j\beta\tau_2}. \tag{96}$$

As with $\mathbf{M_1}$, one can simplify $\mathbf{M_2}$ and $\mathbf{M_3}$ and bring them into a form similar to (95). Using these simplifications for $\mathbf{M_1}$, $\mathbf{M_2}$, and $\mathbf{M_3}$ in (90) we obtain

$$\begin{aligned}\mathcal{C}_{4x}(\alpha;\tau_1,\tau_2,\tau_3) &= \mathcal{M}_{4x}(\alpha;\tau_1,\tau_2,\tau_3) - \sum_{\beta\in\mathcal{A}_2^m}\{\mathcal{M}_{2x}(\alpha-\beta;\tau_1) \\ &\times \mathcal{M}_{2x}(\beta;\tau_3-\tau_2)e^{j\beta\tau_2} \\ &+ \mathcal{M}_{2x}(\alpha-\beta;\tau_2)\mathcal{M}_{2x}(\beta;\tau_1-\tau_3)e^{j\beta\tau_3} \\ &+ \mathcal{M}_{2x}(\alpha-\beta;\tau_3)\mathcal{M}_{2x}(\beta;\tau_2-\tau_1)e^{j\beta\tau_1}\}.\end{aligned} \tag{97}$$

Now (38) follows upon using (35) into (97) for $\mathcal{M}_{kx}, k = 2, 4$.

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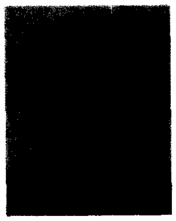

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