MS&E 319/CS 369X: Topics in Network Algorithms. Winter 2005-06
Course URL: `http://www.stanford.edu/~ashishg/network-algorithms`.
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10.1 Shadow Prices, Proportional Fairness, and Stability

Today, we are going to talk about Kelly, et al [1] work. The authors consider the following scenario. Let r be an index for users, and also represent the route for the r^{th} user. Users have different preferences, utility of user r is $U_r(x_r)$ where x_r is the amount of flow she has been assigned, and U_r represents her utility function which supposed to be a strictly concave, non-decreasing, differentiable function. Note that the strictly concaveness condition here is not very important because we know much about linear utility functions.

Our goal is to reach to the optimum solution of the $SYSTEM(U, A, c)$ problem bellow, where U is the vector of utility functions of users, A is the adjacency matrix, (i.e. $A[j, r] = 1 \Leftrightarrow j \in r$), and c is the capacity constraints on the edges.

$SYSTEM(U, A, c)$:

$$maximize \sum_r U_r(x_r) \tag{1}$$

$$Ax \leq c$$
$$x \geq 0$$

While this optimization problem is mathematically tractable (with a strictly concave objective function and a convex region), it involves utilities U that are unlikely to be known by the network. We are thus led to consider two simpler problems. First, consider the Lagrangian of 1:

$$L_1 = \sum_r U_r(x_r) - \sum_j \mu_j (\sum_{r:j\in r} x_r - c_j) \tag{2}$$

Clearly Equation 1 is a Convex optimization problem, so we can write the KKT conditions:

$$\frac{\partial L_1}{\partial x_r} = U_r'(x_r) - \lambda_r = 0, \lambda_r = \sum_{j:j\in r} \mu_j \forall r \tag{3}$$
$$Ax \leq c \tag{4}$$
$$x \geq 0 \tag{5}$$
$$\mu_j \geq 0 \tag{6}$$
$$\mu_j (\sum_{r:j\in r} x_r - c_j) = 0, \forall j \tag{7}$$

The first order condition is then $\frac{\partial L_1}{\partial x_r} = U'_r(x_r) - \lambda_r = 0$, where $\lambda_r = \sum_{j:j\in r}\mu_j$. The complementarity condition is that either $\mu_j = 0$ or $\sum_{r:j\in r} x_r = c_j$. Finding x, μ satisfying these conditions will maximize $\mathcal{U}$.

Kelly's insight is that the first order condition depends only on U_r and λ_r: this is the $USER$ problem. The complementarity condition is just a structural condition: this is the $NETWORK$ problem. This decomposition can simplify analysis of the $SYSTEM$ problem.

Suppose λ_r is the charge per unit for flow r, and are is willing to pay w_r for routing her flow. Then the amount of flow she receives is $x_r = \frac{w_r}{\lambda_r}$. Then the utility maximization problem for user r is:

$USER(U_r, \lambda_r)$:

$$\mathit{maximize}\ U_r(x_r) - \lambda_r x_r \tag{8}$$

$$x_r \geq 0$$

To give you the intuition, note that λ_r are not real prices, they are actually corresponding dual cost, the authors refer to them as *shadow prices*. We will back to this concept soon.

Suppose next that network knows the vector w of all w_rs, and attempts to maximize the function $\sum w_r \log x_r$. The network optimization problem is then as follows.

$NETWORK(A, c, w)$:

$$\mathit{maximize}\ \sum w_r \log x_r \tag{9}$$

$$Ax \leq c$$
$$x_r \geq 0, \forall r$$

The Lagrangian for $NETWORK$ problem is:

$$L_2 = \sum_r \frac{w_r}{x_r} - \sum_r \mu_j(A_j x - cj) \tag{10}$$

So, by KKT conditions, for the optimal solution we have:

$$\frac{\partial L_2}{\partial x_r} = 0 \Rightarrow \frac{w_r}{x_r} = \sum_{j:j\in r}\mu_j = \lambda_r \tag{11}$$

So $USER$ and $NETWORK$ take part to satisfy the conditions 3. So we can take a primal-dual approach. For users, $\lambda_r = \sum_{j:j\in r}\mu_j$ is the dual cost. They change their flow according to this differential equation:

$$\frac{dx_r}{dt} = \lambda_r - U_r'(x_r) \tag{12}$$

From the intuition we have from previous sessions, we know that we should set $\mu_j = f_j(\sum_{r:r\in j} x_r)$ for some increasing steep function f_j. So we can converge to an equilibrium. When we reach to the equilibrium we know that we are in optimum. And Kelly et. al showed that for appropriate choice of f the system will converge to equilibrium. But it is kind of *EE/Theory* works, there is not much about running time analysis, and as far as I know there are no general result when we have discrete updates of flow rate which is what happens in real application. There are some partial results in this area by Ramesh Johari and also Amin Saberi.

10.2 Pricing for Fairness: Distributed Resource Allocation for Multiple Objectives

So far we have seen it is not hard to optimize when we know what we exactly want to maximize over the distribution of resource allocations in network. Garg and Konemann gave an algorithm for problems *maximize* $\sum_r x_r$ or *maximize minimum*$_r$ x_r. Also, TCP extensions like TCP RENO when are we actually maximizing $\sum_r arctg(\frac{1}{1+x^2})$ or TCP VEGAS which approximately solves the problem of *maximize* $\sum_r \log x_r$ can be viewed as especial case of Kelly et. al [1] model.

But there is no universal agreement about what should be our optimization objective. So, this motivates us to find an allocation which approximates all of the desired objective functions.

Let us start to find some common characteristics of the *desired* objective functions. Consider the the following allocations for users A and B in the table bellow:

Allocation	x_A	x_B
P	10	10
Q	5	15
R	15	5
S	15	10

If someone asks which of the allocations Q and R is better, the answer is they are the same, there is no way to distinguish between them. But P, is probably better than both of Q and R. Also S seems to be more efficient that P. Let $U(x_1, x_2, \ldots, x_n)$ be our desired objective function. So, according to these observations, we can say that U should be *symmetric, concave and non-decreasing*. We add additional constraint to U by requiring $U(0) = 0$. We call the family of these functions *Canonical functions*. Our goal is to simultaneously maximize all of the canonical functions.

Consider the following scenario. The are 3 nodes p_1, p_2 and p_3 on a path where p_1, is connected to p_2 and p_2 is connected to p_3 by edges of capacity 1. Also there are 3 flows, x_1 from p_1 to p_2, x_2 from p_2 to p_3, x_3 from p_1 to p_3. Then the solution for *maximize* $x_1 + x_2 + x_3$ is 2, for $x_1 = x_2 = 1$ and $x_3 = 0$. But, the solution for *maximize minimum*$\{x_1, x_2, x_3\}$ is $\frac{1}{2}$, for $x_1 = x_2 = x_3 = \frac{1}{2}$. So we can not find a solution which maximize of all the canonical functions but we may hope for an approximation solution.

Let first define $P_j(x)$ as the some of the k-th minimum element in vector x, e.g. P_1 is the $min_i\ x_i$

and P_n is the sum over all x_is. This is a theorem by famous mathematicians Hardy, Littlewoods and Polya:

Proposition 10.1 *$U(x) \geq U(y)$ for all symmetric concave U iff*

- $\sum_1^n x_i = \sum_1^n y_i$
- $P_j(x) \geq P_j(y), 1 \leq j \leq n$

If x and y satisfy the conditions above we say x is majorized by y which is denoted by $x \preceq y$, the notation may be confusing because in the original theorem they have considered the functions convex. As we see there is no global minimum so we need the following variation of the theorem:

Theorem 10.2 *$U(x) \geq \frac{U(y)}{\alpha}$, for all canonical function U iff:*

$$P_j(x) \geq P_j(y), \forall j$$

In some sense we can say P_js are *complete* for class of canonical utility functions. We prove that P_js are concave:

Lemma 10.3 *P_js are concave.*

Proof: Let $z = \beta z + (1-\beta)y, 0 \leq \beta \leq 1$, and let $\sigma(i)$ be the index of the i^{th} smallest component of z.

$$P_j(z) = \sum_{i=1}^{j} z_{\sigma(i)} = \beta \sum_{i=1}^{j} x_{\sigma(i)} + (1-\beta) \sum_{i=1}^{j} y_{\sigma(i)} \geq \beta P_j(x) + (1-\beta) P_j(y)$$

■

As the very first step, lets find a $n-approximation$ solution for all P_js. Consider this example, we have three flows, x_1, x_2 and x_3 with the capacity constraint: $x_1 + 20x_2 + 500x_3 \leq 1$. Then

$maxP_1 : x_1 = x_2 = x_3 = \frac{1}{521}$

$maxP_2 : x_1 = x_2 = \frac{1}{21}, x_3 = 0$

$maxP_3 : x_1 = 1, x_2 = x_3 = 0$

It is clear if we average all of this solutions the result will be at least $\frac{1}{n}$ of each of P_js. Next session we will complete our discussion about this topic.

References

[1] F. Kelly, A. Maulloo, and D. Tan. *Rate control in communication networks: shadow prices, proportional fairness and stability.* Journal of the Operational Research Society, 49:237–252, 1998.

[2] S. Cho, A. Goel. *Pricing for Fairness: Distributed Resource Allocation for Multiple Objectives.* To be appeared in STOC 2006, 49:237–252, 1998.