

Using Game Theory to Analyze Physical Layer Cognitive Radio Algorithms

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1. Introduction

While considerable work has gone into developing and demonstrating the feasibility of cognitive radio, the vast majority of this work has focused on demonstrating the gains a single link can achieve when adapting to an interference environment. While the results from this work are important, the underlying assumptions are not a realistic because the modeling of the environment and the operational scenarios fail to consider the existence of a major source of interference – other cognitive radios.

Unlike traditional interferers, cognitive radios adapt their operation in response to their perceived interference environment. When numerous cognitive radios are collocated, this interference environment may be constantly changing as the cognitive radios adapt to the other cognitive radios adaptations. Because of this recursive process, serious concerns are introduced:

- Under what conditions will the recursions settle down to a steady state?
- What is that steady-state?
- Will the resources be hoarded by a single radio/link or will they be equitably shared among the radios?
- Will the cognitive radios actually make use of available spectrum without impinging on other radios' spectrum rights?
- How much bandwidth will be consumed with signaling overhead and how much bandwidth will actually be used for data transfer?

Game theory, a collection of mathematical models and techniques for the analysis of interactive decision processes, is particularly well suited for answering these questions. In prior work we developed a set of game theoretic tools for analyzing the interactive adaptations of numerous cognitive radio algorithms and a currently applying this work to the design of physical layer cognitive radio algorithms.

This paper describes the analytical insights we have gained in cognitive radio physical layer algorithms, highlighting techniques for identifying steady states, determining the kinds of adaptations that are assured of convergence, and establishing stability when imperfect information is present. Based on techniques described in this discussion, the paper then describes our ongoing work applying our insights to algorithm design for cognitive radios.

2. Cognitive radio and game theory

This section provides a brief review of cognitive radio, game theory and the application of game theory to cognitive radio.

2.1 Cognitive radio

Cognitive radios are adaptive radios that are aware of their capabilities, aware of their environment, aware of their intended use, and able to learn from experience new waveforms, new models, and new operational scenarios. Numerous regulatory bodies such as the FCC [1] and ITU [2] have expressed interest in cognitive radios software radios as a way to create additional usable spectrum by allowing radios to dynamically respond to unused spectrum.

Since being introduced by Mitola [3], the operation of cognitive radios has been frequently envisioned by the cognition cycle shown in Figure 1. The cognition cycle is a state machine that resides in the cognitive radio and defines how the radio learns about and reacts to its operating environment.

In the cognition cycle, a radio receives information about its operating environment (**Outside world**) through direct observation or through signaling. This information is then evaluated (**Orient**) to determine its importance. Based on this valuation, the radio determines its alternatives (**Plan**) and chooses an alternative (**Decide**) in a way that presumably would improve the valuation. Assuming a waveform change was deemed necessary, the radio then implements the alternative (**Act**) by adjusting its resources and performing the appropriate signaling. These changes are then reflected in the interference profile presented by the cognitive radio in the **Outside world**. Throughout the process, the radio is using these observations and decisions to improve the operation of the radio (**Learn**), perhaps by creating new modeling states, generating new alternatives, or creating new valuations.

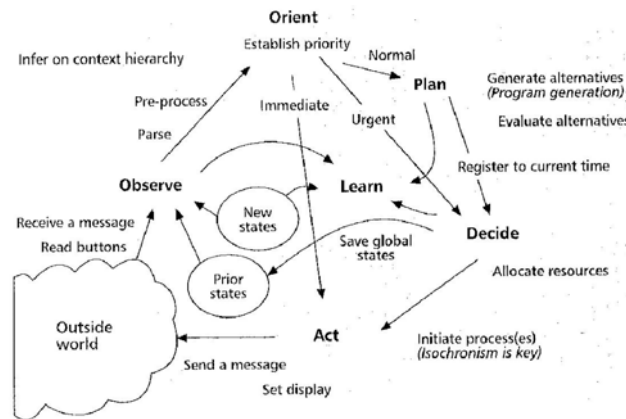


Figure 1 Cognition Cycle [3]

2.2 Game theory

Game theory is a set of mathematical tools used to model and analyze interactive decision processes. The fundamental component of game theory is the notion of a game. When expressed in *normal form*, a game, $G = \langle N, A, \{u_i\} \rangle$, has the following three primary components.

1. A finite set of players (decision makers) typically denoted $N = \{1, 2, \dots, n\}$.
2. An action space, A , formed from the Cartesian product of each player's action set, $A = A_1 \times A_2 \times \dots \times A_n$.

3. A set of utility functions, $\{u_i\} = \{u_1, u_2, \dots, u_n\}$, that quantify the players' preferences over the game's possible outcomes. Outcomes are determined by the particular action chosen by player i , a_i , and the particular actions chosen by all of the other players in the game, a_{-i} .

In the game, players are assumed to act in their own self-interest, that is to say, each player chooses its actions in such a way that increases the number returned from its utility function. Typically normal form games are analyzed to identify steady-states known as Nash equilibria. A particular action tuple, a^* in A , is said to be a Nash equilibrium if no player can improve its payoff, $u_i(a^*)$, by unilaterally changing its action.

Another typical game model is the *repeated game* model. A repeated game is sequence of stages where each stage is the same normal form game. A repeated game is fully characterized by a stage game, a *player function* that defines which players are allowed to adapt play in that stage, and a *set of decision rules* that describe the rules that each player follows to update its decisions when it is that player's turn to play.

For repeated games, it makes sense to discuss concepts such as *convergence* and *stability*. Given a repeated game where a^k represents the action tuple played in stage k , a sequence $\{a^k\}$ in action space A , $A \in \square^{n \times m}$ is said to converge if there is an action tuple a^* such that for every $\varepsilon > 0$, there is an integer K such that $k \geq K$ implies $\|a^k, a^*\| < \varepsilon$. Similarly, a^* is said to be *Lyapunov stable* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $k \geq 0$, $\|a^0, a^*\| < \delta \Rightarrow \|a^k, a^*\| < \varepsilon$. Showing that a fixed point is Lyapunov stable is normally accomplished by finding a Lyapunov function for the region around the fixed point.

Identifying the conditions under which sequences in a repeated game converge or are stable often require the introduction of special game models. Two such models are given Section 3.

2.3 Applying game theory to cognitive radio

Examining again the cognition cycle shown in Figure 1, it is readily seen how the interactions of a network of cognitive (or adaptive) radios maps into a game.

Each node in the network that implements the decision step (making it a decision maker) of the cognition cycle is a player in the game. The various alternatives available to a node forms the node's action set, and the action space is formed from the Cartesian product of the radios' alternatives. A cognitive radio's observation and orientation steps combine to form a player's utility function. Loosely, the observation step provides the player with the arguments to evaluate the utility function, and the orientation step determines the valuation of the utility function.

Note that we have ignored the learning step of the cognition cycle. This is not an oversight nor indicative of a limitation of game theory. Rather, it is a limitation of the normal form game model. While the repeated game with a normal form stage game is

appropriate for any adaptive radio algorithm or for any cognitive radio adaptations that do not require learning, it is not appropriate for analyzing algorithms that learn. In this case, more advanced game models that incorporate learning processes, such as *Bayesian games* should be used. It should also be noted that game theory is not well suited to games where actions and objectives are not well defined as may be the case when cognitive radios learn over time.

There are five questions that game theory should answer when analyzing an adaptive algorithm:

1. Does the algorithm have a steady state?
2. What are those steady states?
3. Is the steady state(s) desirable?
4. What restrictions need to be placed on the decision update algorithm to ensure convergence?
5. Is the steady state(s) stable?

While questions 1-3 can be addressed through traditional game theory techniques, questions 4 and 5 require additional information that can be provided through the introduction of certain game models. Previously [4], we have argued that determining if a steady-state is desirable is best determined by showing the the steady-state maximizes some global network objective function.

3. Relevant Game Models

This section reviews two valuable game models: the supermodular game model and the potential game model and examines how each of these game models address questions 1,2, 4, and 5 considered in Section 2. For all models, it is preferable address the third question by substituting the predicted network steady state(s) into a network objective function.

3.1 Potential game model

A potential game is a special normal form game where there is a function, $V : A \rightarrow \mathbb{R}$, such that when a unilateral deviation occurs, the change in V , ΔV , is reflected in the change in value seen by the unilaterally deviating player, Δu_i . If for all unilateral deviations, $\Delta V = \Delta u_i$ the game is called an exact potential game; likewise if $\text{sgn}(\Delta V) = \text{sgn}(\Delta u_i)$ the game is an ordinal potential game.

Model Identification: A game can be shown to be an exact potential game if the action space is compact and the utility functions satisfy (1).

$$\frac{\partial^2 u_i(a)}{\partial a_i \partial a_j} = \frac{\partial^2 u_j(a)}{\partial a_j \partial a_i} \forall i, j \in N, a \in A \quad (1)$$

Other than applying the definition, there is no well-defined condition for verifying that a game is an ordinal potential game. However, [5] shows that if a sequence of ordinal (monotonic) transformations of the utility functions result in an exact potential game, then the original game is an ordinal potential game.

In addition to the potential game models discussed here, a large number of potential game models exist with differing relations between Δu_i and ΔV and differing identification techniques [5]. discusses these game models in great depth.

Steady-state Existence: Potential games with a compact action space always have at least one steady-state [6].

NE Identification: All maximizers of V are NE [6]. Note this need not be all of the NE in the game, but the only stable NE in the game are maximizers of V .

Convergence: Potential games have the finite improvement path (FIP) property, so when nodes act in a selfish manner play converges to a NE.

Stability: For repeated games where the decision rules result in nonincreasing utility for the deviating players, the potential function is a Lyapunov function [5].

3.2 Supermodular game model

A game is termed supermodular if the action space forms a *lattice* and the utility functions are *supermodular*. A partially ordered set, X , is termed a lattice if for all $a, b \in X$, $a \wedge b \in X$ and $a \vee b \in X$ where $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$. A function $f : X \rightarrow \mathbb{R}$ where X is a lattice, is termed supermodular if for all $a, b \in X$, $f(a) + f(b) \leq f(a \wedge b) + f(a \vee b)$.

Model Identification: While the definition may seem complicated, a game can be identified as a supermodular game if all players' utility functions satisfy the relationship given in (2) and the action space is compact.

$$\frac{\partial^2 u_i(a)}{\partial a_i \partial a_j} \geq 0 \forall j \neq i \in N \quad (2)$$

NE Existence: By Topkis's fixed point theorem [7], all supermodular games have at least one NE.

NE Identification: By [7], all NE for a game form a lattice. While this does not particularly aid in the process of initially identifying NE, from every pair of identified NE, e.g., a^* and b^* , additional NE can be found by evaluating $a^* \wedge b^*$ and $a^* \vee b^*$.

Convergence: By [8], supermodular games have weak FIP, i.e., from any initial action vector, there exists a sequence of selfish adaptations that lead to a NE. Specifically for supermodular games, when the decision rules for all players are best responses, play will converge to a NE [8]. Further, if the radios make a limited number of errors or if the radios are instead playing a best response to a weighted average of observations from the recent past, play will converge [8][9]. These same convergence results also hold for potential games as FIP implies weak FIP.

Stability: Supermodular games are a subset of generalized best response potential games [10]. For generalized best response potential games with a finite action space and continuous utility functions the following is a Lyapunov function. Define $\sigma(a)$ as the set of action tuples for which a is the best response for some player, or more formally, $\sigma(a) = \{a' \in A : a \in \hat{B}(a')\}$. Then $V(a) = |\sigma(a)| + \sum_{a' \in \sigma(a)} V(a')$ is a Lyapunov function for the game [10]. Note that identifying the maximizers of this V also identifies the stable steady states for the game.

4. Sample Applications

The following present a number of simple examples of physical layer algorithms that can be shown to be a potential game or a supermodular game and thus can be assured of a steady-state, convergence to that steady-state, and stability.

4.1 Frequency Selection Adaptive Interference Avoidance

Consider an ad-hoc network of cognitive radio links operating in master-slave fashion. Each link, j , is implementing a waveform with bandwidth B and center frequency f_j . The master node on each link j directs the link to adjust f_j so that the interference that link j is minimized. To simplify this example, we'll assume all links are transmitting at a fixed power level and there is symmetric path loss between links.

A single iteration of this game can be expressed as a normal form game as follows The player set is the set of links, N . Each player's action set is the set of frequencies, F , available to that link. A utility function for any player, j , is given by $u_j(f) = \sum_{k \in N \setminus j} \sigma(f_j, f_k)$ where $\sigma(f_j, f_k) = \min\{|f_j - f_k|, B\}$. This game is an exact

potential game with a potential function given by $V(f) = \sum_{j=1}^{|N|} \sum_{j+1}^{|N|} \sigma(f_j, f_k)$.

Thus this network is assured of having a steady-state which can be identified by solving for the maximizers of $V(a)$, is assured of converging assuming each link acts in its own interests and the link allowed to adapt at any particular time is chosen randomly, and is stable.

Shown below in Figure 2 is the output of a simulation of this system where there are 10 links, $B = 1$ MHz, $F = [0, 10]$ MHz, and each master node chooses the frequency that maximizes its utility. Figure 3 shows another realization of this simulation. Note that while the existence of a steady state, convergence to a steady state, and stability of any steady-state is assured by virtue of being a potential game, there are actually numerous steady-states in this network, not all of which achieve the optimal channel spacing. Also note that since there are numerous fixed points, no fixed point is globally stable.

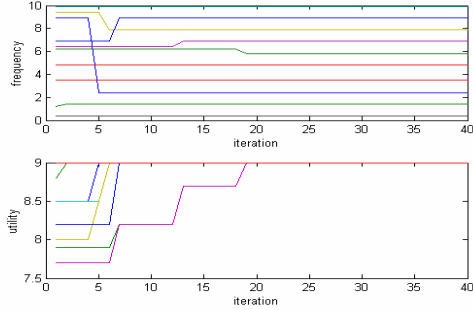


Figure 2 Frequency Selection AIA

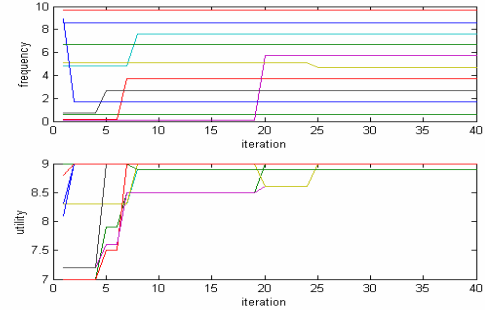


Figure 3 Frequency Selection AIA

4.2 OFDM Channel Filling

Consider a closely spaced ad-hoc network of cognitive radio links operating in master-slave fashion. Each link, j , has a number of channels, C over which it can transmit and chooses an action a_j that corresponds to a choice of zero to many channels to simultaneously operate on. The benefit that each node gets from transmitting on a particular channel, c , is given by $f_c(\sigma_c(a))$ where $\sigma_c(a)$ returns the number of links simultaneously operating on channel c given that the radios are playing action tuple a .

A single iteration of this game can be expressed as a normal form game as follows. The player set, N , is given by the set of master nodes. The action set of each player is given by the power set of the channel set, 2^C . A utility function for this any player, j , is given by $u_j(a) = \sum_{c \in a_j} f_c(\sigma_c(a))$. This game is an exact potential game with a potential function

$$\text{given by } V(a) = \sum_{c \in \bigcup_{i=1}^n a_i} \left(\sum_{k=1}^{\sigma_c(a)} f_c(k) \right).$$

Thus this network is assured of having a steady-state which can be identified by solving for the maximizers of $V(a)$, is assured of converging assuming each link acts in its own interests and the link allowed to adapt at any particular time is chosen randomly, and is stable. Note as there are numerous steady-states for this game, no steady-state is globally stable, though each steady-state is locally stable.

4.3 Distributed Power Control

Now consider an ad-hoc network of cognitive radio links operating in master-slave fashion. All links are operating on the same channel using a waveform that has spreading factor K . Each master node, j , has power levels $P_j = [0, P^{\max}]$ and directs the link to change transmit power level in an attempt to achieve a target SINR, γ_j . increase is implementing a waveform with bandwidth B and center frequency f_j .

A single iteration of this game can be expressed as a normal form game as follow. The player set, N , is given by the set of master nodes. The action set of each player is given by its set of power, P_j . A utility function for this any player, j , is given by

$u_j(p) = \left(\gamma_j - \log(h_{jj}p_j) + \log\left(\frac{1}{K} \sum_{k \in N \setminus j} h_{kj}p_k + N_0\right) \right)^2$ where h_{jk} is a nonnegative scalar

representing the fraction of power transmitted by link j that is actually received by link k . This game can be identified as a supermodular game as follows.

$$\frac{\partial u_j(p)}{\partial p_j} = 2 \left(\gamma_j - \log(h_{jj}p_j) + \log\left(\frac{1}{K} \sum_{k \in N \setminus j} h_{kj}p_k + N_0\right) \right) \frac{1}{p_j} \log(e).$$

$$\frac{\partial^2 u_j(p)}{\partial p_j \partial p_k} = -\frac{\frac{1}{K} h_{kj}}{p_j \left(\frac{1}{K} \sum_{k \in N \setminus j} h_{kj}p_k + N_0 \right)} \log(2e) > 0.$$

Thus this network is assured of having a steady-state and is assured of converging assuming each link acts in its own locally optimal manner.

A simulation was constructed for a two cluster network with 11 nodes, $K=63$, and a path loss exponent of 4 was constructed where each link has a target SINR of 8.4 dB. A noiseless version of this simulation is shown in Figure 4 and a noisy version is shown in Figure 5. Note that the noisy simulation implies that the system is also stable.

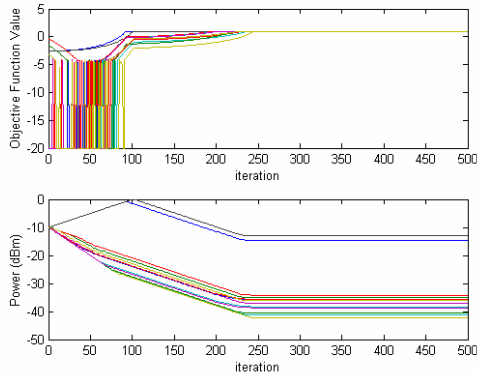


Figure 4 Noiseless Simulation

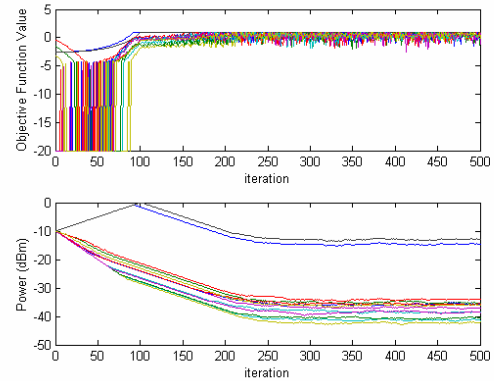


Figure 5 Noisy Simulation

5. Summary

We have shown how the cognition cycle maps into a normal form game model and identified five issues that any application of game theory to cognitive or adaptive radio should address: steady state existence, steady state identification, steady-state optimality, convergence, and stability. To address these issues, we have adopted a model-based approach to analyze cognitive radio physical layer algorithms. Using potential and supermodular game models, we can readily identify when a cognitive radio algorithm has a steady state, determine the kinds of adaptations that are assured of convergence, and establish stability regions. We then reviewed a few different physical layer algorithms and used these game models to assure the existence of a steady-state, convergence, and stability. We then verified these theoretical results through simulation.

6. References

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